

Three loop \overline{MS} operator correlation functions for deep inelastic scattering in the chiral limit

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

JHEP04(2009)127

(<http://iopscience.iop.org/1126-6708/2009/04/127>)

[The Table of Contents](#) and [more related content](#) is available

Download details:

IP Address: 80.92.225.132

The article was downloaded on 03/04/2010 at 10:30

Please note that [terms and conditions apply](#).

Three loop $\overline{\text{MS}}$ operator correlation functions for deep inelastic scattering in the chiral limit

J.A. Gracey

*Theoretical Physics Division, Department of Mathematical Sciences, University of Liverpool,
P.O. Box 147, Liverpool, L69 3BX, U.K.*

E-mail: gracey@liv.ac.uk

ABSTRACT: We compute a variety of operator-operator correlation functions to third order in the $\overline{\text{MS}}$ scheme in the chiral limit. These include combinations of quark bilinear currents with gauge invariant operators such as moments $n = 2$ and 3 of the flavour non-singlet Wilson operators of deep inelastic scattering and moment $n = 2$ of the transversity operator, as well as the correlation functions of the latter operators with themselves. The explicit values of these gauge independent correlation functions are required to assist with non-perturbative matching to lattice regularized calculations of the same quantities. As part of the computation we determine the mixing matrix of renormalization constants of these non-singlet currents with their associated total derivative operators at the same twist to three loops in $\overline{\text{MS}}$. Such operators are crucial in extracting renormalization constants for the operator-operator correlation functions which are consistent with the corresponding renormalization group equation. As a by-product we deduce the R -ratio for the tensor current to third order in the $\overline{\text{MS}}$ scheme.

KEYWORDS: NLO Computations, Renormalization Group, QCD, Deep Inelastic Scattering

Contents

1	Introduction	2
2	Preliminaries	4
3	Renormalization group	9
4	Results	19
4.1	Scalar-scalar	20
4.2	Vector-vector	20
4.3	Tensor-tensor	21
4.4	Vector-Wilson 2	21
4.5	Wilson 2-Wilson 2	22
4.6	Vector-Wilson 3	24
4.7	Wilson 3-Wilson 3	24
4.8	Tensor-transversity 2	30
5	Tensor current R-ratio	32
6	Discussion	33
A	Projectors	33
A.1	Vector-vector	34
A.2	Tensor-tensor	34
A.3	Vector-Wilson 2	34
A.4	Vector-Wilson 3	35
A.5	Wilson 2-Wilson 2	35
A.6	Wilson 3-Wilson 3	35
A.7	Tensor-transversity 2	37
B	Expressions for $SU(3)$	38
B.1	Vector currents	38
B.2	Wilson moment $n = 2$	38
B.3	Wilson moment $n = 3$	39
B.4	Transversity moment $n = 2$	40

1 Introduction

Lattice regularization of non-abelian gauge theories has provided many insights into the non-perturbative properties of the quantum field theory underlying the strong force, known as Quantum Chromodynamics (QCD). For instance, now that dynamical quarks can be treated via the use of powerful supercomputers, the meson and hadron spectra are on the whole in remarkable agreement with experimental data. One current problem of interest for lattice regularization is the measurement of Green's functions relevant for, say, deep inelastic scattering. For instance, whilst the perturbative renormalization of the underlying twist-2 flavour non-singlet and singlet operators are known to three loop accuracy in the $\overline{\text{MS}}$ scheme, [1–7], the associated matrix elements are also required but they can only be fully computed using non-perturbative methods. These are crucial to fully understanding the structure functions of the original hadrons and mesons which are broken up in deep inelastic experiments. Such matrix elements are, however, accessible via lattice regularization with notable progress through the years via collaborations such as QCDSF, [8–13], and others, [14–17]. However, such measurements of Green's functions on the lattice need a variety of techniques to allow comparison with continuum results. Therefore, we will first briefly discuss some of the relevant issues in a general context before formulating the aims for the current article.

The first issue is that the lattice computations necessarily renormalize their operators and Green's functions using a renormalization scheme which is not the standard $\overline{\text{MS}}$ one. The general name for the scheme we refer to in this context is regularization invariant (RI), [18]. Though in practice the main scheme used is referred to as RI'. (In some articles this is synonymous with RI-MOM.) Therefore, one needs a means to convert lattice results from RI type renormalization schemes to the reference scheme of $\overline{\text{MS}}$, [18, 19]. For certain classes of Green's functions the continuum definition and use of RI' (and RI) have been given in three and four loop renormalization of (massless) QCD in arbitrary covariant gauges, [20–23]. The second main issue to deal with is that of matching lattice regularized results for the Green's functions with the corresponding continuum results. There are two main approaches for this. The first is the Schrödinger functional method, [24, 25], upon which we make no further comment. The second is the matching to explicit perturbative QCD results for the *same* Green's function which is what we concentrate on here. In this approach, [18–23], the main ethos is to compute the Green's function to as high a loop order as possible in perturbation theory as a function of the gauge coupling constant, g . Then the lattice computation should match on to the continuum behaviour in the *same* renormalization scheme in the high energy limit. Having a perturbative result to as high a loop order as is computationally feasible will in principle lead to a more accurate extraction of numerical values for the Green's function in the non-perturbative range of interest for, say, nucleon structure functions.

As already indicated, previous work in the continuum concentrated on a variety of gauge invariant twist-2 flavour non-singlet operators, \mathcal{O} , and the perturbative evaluation of the flavour non-singlet Green's function $\langle \psi(p)\mathcal{O}(0)\bar{\psi}(-p) \rangle$ in the chiral limit, [20–23], where p is the momentum flowing through the Green's function and ψ is the quark field.

The operators considered for this Green's function were the Wilson and the transversity operators to and including moment $n = 3$ and 4 respectively and various quark bilinear operators such as the tensor current, [21–23]. However, whilst such results were useful in many ways, they suffer from one major drawback. This is simply stated by noting that although each of the operators considered was gauge invariant, the Green's function itself was gauge dependent. Although ultimately one was only interested in the Landau gauge, the results of [21–23] were provided in an arbitrary linear covariant gauge. Whilst this was not too problematic for continuum calculations, from the point of view of lattice regularization one has also to fix the Landau gauge. However, this is a tough exercise in itself and in principle could open up reliability issues to do with say ensuring the Gribov problem was avoided. To circumvent these lattice regularization gauge fixing issues another approach has been devised.¹ Briefly to extract the appropriate renormalization constants for the operators and hence determine the finite parts of the Green's functions, the approach is to consider gauge independent correlation functions of gauge invariant operators. In this way the potential gauge fixing ambiguity never becomes an issue in the first place since the gauge does not then need to be fixed on the lattice. More specifically the appropriate correlation function to consider is $\langle \mathcal{O}(p)\mathcal{O}(-p) \rangle$. However, for, say, high moment Wilson operators the increase in the number of covariant derivatives may lead to too noisy a numerical signal for extraction of meaningful values of the Green's function. Hence, rather than consider this diagonal correlation function, a simple proposal would be to analyse an off-diagonal correlator such as $\langle \mathcal{O}^1(p)\mathcal{O}^2(-p) \rangle$ where $\mathcal{O}^1(p)$ is a Wilson operator, say, and $\mathcal{O}^2(p)$ is a simple quark bilinear current operator. This has to be chosen in such a way that the Green's function is not simply trivial in the chiral limit. However, information on the Wilson operator renormalization constant can still be derived.

Having reviewed the background and key issues we now indicate the main aim of this article. It is to simply to provide the explicit values of the appropriate operator correlation functions, $\langle \mathcal{O}^1(p)\mathcal{O}^2(-p) \rangle$, relevant to the lattice problem to as many loop orders in perturbation theory that is computationally feasible. Specifically we will focus on three sets of flavour non-singlet operators used in deep inelastic scattering. These are the Wilson operators with moments 2 and 3 and moment 2 of the set of transversity operators. Several correlations of the operator with itself will be provided as well as the appropriate non-zero off-diagonal one. We will present results to third order or three loops in the $\overline{\text{MS}}$ scheme as a function of the momentum flowing through the 2-point function. We note that whilst we compute three loop Feynman diagrams, since it is clear that the leading diagram of the correlator is independent of the strong coupling constant, a , then the results will be to $O(a^2)$ inclusive where $a = g^2/(16\pi^2)$. However, it will also become evident that it is not possible to consider correlators of higher moment operators and expect to evaluate the correlation functions to the same three loop order. Whilst our main motivation is to provide the finite parts of these correlation functions several technical issues need to be addressed to obtain the correct answers. For instance, there is an assumption that the

¹The author is grateful to Dr P.E.L. Rakow and Dr R. Horsley for their patient enlightenment on this point.

flavour non-singlet Wilson and transversity operators do not mix under renormalization. It will turn out that this observation needs to be clarified within the present context. Their three loop $\overline{\text{MS}}$ anomalous dimensions are available, [1–7], and the lower loop results were originally obtained by considering the renormalization of $\langle\psi(p)\mathcal{O}(0)\bar{\psi}(-p)\rangle$. In this momentum configuration the mixing is not relevant. However, in the momentum configuration for the correlators of the present article $\langle\mathcal{O}^1(p)\mathcal{O}^2(-p)\rangle$, since a momentum flows through the operator the mixing is relevant and *cannot* be neglected. Suffice to say at this point that the additional operators are gauge invariant but total derivatives. Therefore, as part of our correlator renormalization programme we have had to compute the relevant anomalous dimension mixing matrices to allow us to extract consistently renormalized correlation functions. Such results will no doubt be important for other areas of deep inelastic scattering such as generalized parton distribution function analyses.

The article is organised as follows. Section 2 introduces the notation, operators and general features of the correlation functions we consider throughout. The general renormalization properties of the underlying correlation functions are discussed in section 3 including the operator mixing issue. Section 4 is devoted to the very mundane but important exercise of recording all the results for the Green’s functions we have considered. As a spin-off we record the R -ratio for the tensor current to third order in section 5. Finally, after concluding remarks in section 6, several appendices are provided. The first records the Lorentz tensor decomposition of several operator correlators. This is necessary since the lattice requires the use of operators with uncontracted indices. In the renormalization of the matrix elements $\langle\psi(p)\mathcal{O}(0)\bar{\psi}(-p)\rangle$ of [1–3], the Lorentz indices were contracted with a null vector Δ_μ with $\Delta^2 = 0$. This was because, for example, the Wilson operators were traceless and symmetric and this contraction excluded the part with metric tensors in order to ease the extraction of renormalization constants directly. Here, since we will use a very specific computer algebra package and algorithm which only operates on scalar Feynman integrals, we need to project out the relevant scalar amplitudes with respect to some tensor basis, which is, of course, not unique. This is discussed in appendix A. The remaining appendix records the explicit *numerical* values of the various finite parts of the correlation functions for the colour group SU(3) which were originally presented in exact form to $O(a^2)$ in section 4.

2 Preliminaries

In this section we define our notation and operators and discuss the operator correlation functions from a general perspective. Throughout we use the standard QCD Lagrangian with massless quarks to immediately put us in the chiral case and an arbitrary linear covariant gauge fixing which is parametrized by the (renormalized) parameter α . However, since our correlation functions involve gauge invariant operators and therefore are gauge independent, α will never actually appear in any of our final correlator expressions. Though we stress that at no stage have we set $\alpha = 0$ internally in our computations. Its *natural* cancellation is a strong internal consistency check on the construction of, say, our Feynman rules and the operator renormalization. Given this we define the general correlation

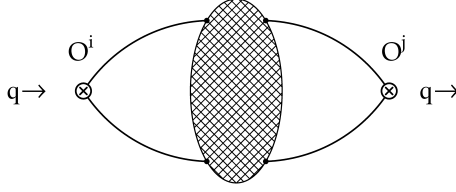


Figure 1. Operator correlation function $\langle \mathcal{O}^i(q)\mathcal{O}^j(-q) \rangle$.

function as, [26–28],

$$\Pi_{\mu_1 \dots \mu_{n_i} \nu_1 \dots \nu_{n_j}}^{ij}(q^2) = (4\pi)^2 i \int d^d x e^{iqx} \langle 0 | \mathcal{O}_{\mu_1 \dots \mu_{n_i}}^i(x) \mathcal{O}_{\nu_1 \dots \nu_{n_j}}^j(0) | 0 \rangle \quad (2.1)$$

where q is the momentum (with $q^2 = -Q^2$) and we have labelled the Lorentz indices of the respective constituent operators by a different parent Greek letter for clarity. At this point it is worth noting that we closely follow the procedures of [26–28] which are excellent reviews of calculating (2.1) for quark current operators. The Green’s function itself is illustrated schematically in figure 1 with the momentum flow made explicit. Our notation needs explanation. We use superscripts i and j to denote the left and right operators \mathcal{O} of the correlation function as indicated in figure 1 where the momentum flows into the left operator and out through the right operator. In principle, these operators are different whence the two distinct sets of Lorentz indices $\{\mu_i\}$ and $\{\nu_j\}$. Though for some cases we will include the quark mass operator which has no Lorentz tensor structure and so no such indices will be formally required. The flavour indices have been omitted to avoid cluttering the notation further. It is understood that there is a flavour generator included within each operator and later we will note the internal accounting method used to ensure that we derive results only for the correlation of flavour non-singlet currents as opposed to flavour singlet currents. For the latter there would be an additional but different operator mixing problem from that into total derivative operators. This singlet mixing is already well documented, [1–3]. In other words flavour singlet quark bilinear current operators can mix into purely gluonic operators with the same twist and quantum numbers, as well as gauge variant operators with the same properties but constructed from, say, Faddeev-Popov ghost fields. Moreover, one would also have to handle equation of motion operators too. We mention this aspect for completeness and note that as far as we are aware there is currently no lattice proposal to examine the flavour singlet case and we therefore will not devote any time to it here in the analogous continuum problem.

Since we will consider a variety of operators we introduce a shorthand notation, akin to [26–28], for the superscripts i and j to indicate which operator appears in (2.1). These are listed below as

$$\begin{aligned} S &\equiv \bar{\psi}\psi \\ V &\equiv \bar{\psi}\gamma^\mu\psi \\ T &\equiv \bar{\psi}\sigma^{\mu\nu}\psi \\ W_2 &\equiv \mathcal{S}\bar{\psi}\gamma^\mu D^\nu\psi \end{aligned}$$

$$\begin{aligned}
 \partial W_2 &\equiv \mathcal{S} \partial^\mu (\bar{\psi} \gamma^\nu \psi) \\
 W_3 &\equiv \mathcal{S} \bar{\psi} \gamma^\mu D^\nu D^\sigma \psi \\
 \partial W_3 &\equiv \mathcal{S} \partial^\mu (\bar{\psi} \gamma^\nu D^\sigma \psi) \\
 \partial \partial W_3 &\equiv \mathcal{S} \partial^\mu \partial^\nu (\bar{\psi} \gamma^\sigma \psi) \\
 T_2 &\equiv \mathcal{S} \bar{\psi} \sigma^{\mu\nu} D^\sigma \psi \\
 \partial T_2 &\equiv \mathcal{S} \partial^\mu (\bar{\psi} \sigma^{\nu\sigma} \psi)
 \end{aligned} \tag{2.2}$$

where \mathcal{S} denotes the appropriate symmetrization in the Lorentz indices as well as the operator's tracelessness which we will discuss shortly. This list already reveals our hand in terms of the operator mixing issue. Moreover, in choosing this notation we have disguised to a degree what some of the total derivative operators are. However, we choose to work within certain sectors. By this we mean, using the Wilson operator of moment 3 for illustration, that W_3 is the main label for that sector as well as being the parent operator. In its renormalization it spawns offspring total derivative operators denoted by one or more ∂ symbols. These are labelled ∂W_3 and $\partial \partial W_3$. However, clearly from (2.2) these derive from the parent W_2 operator of the preceding or lower sector and the vector current of the first sector. It therefore might have been apt to choose the respective notation for these to be ∂W_2 and $\partial \partial V$ for labels. We have chosen not to do so because for the W_3 sector the number of Lorentz indices are the same on each combination of these operators. Moreover, one would have to count the derivative labels to deduce which sector the label was associated with. Hence the Lorentz projection tensors used to project out the scalar amplitudes we will compute are the same for these combinations. So for the W_3 sector there is only one basic projector. Whilst this means we will use W_3 as both an operator label and sector indicator, this we believe will minimize the confusion in choice of notation when trying to ascertain which sector, say, ∂V belongs to especially when transversity is dealt with in parallel calculations.

Next in (2.2) we choose the symmetrization with respect to Lorentz indices and the tracelessness conditions in the standard way. For transversity, which involves the γ -matrix commutator $\sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$, the definition is not the same as for the Wilson operators, [29–31]. For the three moments we consider, the explicit definitions of the symmetric traceless operators are, in d -dimensions,

$$\begin{aligned}
 \mathcal{S} \mathcal{O}_{\mu\nu}^{W_2} &= \mathcal{O}_{\mu\nu}^{W_2} + \mathcal{O}_{\nu\mu}^{W_2} - \frac{2}{d} \eta_{\mu\nu} \mathcal{O}_\sigma^{W_2} \\
 \mathcal{S} \mathcal{O}_{\mu\nu\sigma}^{W_3} &= \mathcal{O}_{S\mu\nu\sigma}^{W_3} - \frac{1}{(d+2)} \left[\eta_{\mu\nu} \mathcal{O}_{S\sigma\rho}^{W_3} + \eta_{\nu\sigma} \mathcal{O}_{S\mu\rho}^{W_3} + \eta_{\sigma\mu} \mathcal{O}_{S\nu\rho}^{W_3} \right] \\
 \mathcal{O}_{S\mu\nu\sigma}^{W_3} &= \frac{1}{6} \left[\mathcal{O}_{\mu\nu\sigma}^{W_3} + \mathcal{O}_{\nu\sigma\mu}^{W_3} + \mathcal{O}_{\sigma\mu\nu}^{W_3} + \mathcal{O}_{\mu\sigma\nu}^{W_3} + \mathcal{O}_{\sigma\nu\mu}^{W_3} + \mathcal{O}_{\nu\mu\sigma}^{W_3} \right] \\
 \mathcal{S} \mathcal{O}_{\mu\nu\sigma}^{T_2} &= \mathcal{O}_{\mu\nu\sigma}^{T_2} + \mathcal{O}_{\mu\sigma\nu}^{T_2} - \frac{2}{(d-1)} \eta_{\nu\sigma} \mathcal{O}_{\mu\rho}^{T_2} + \frac{1}{(d-1)} \left[\eta_{\mu\nu} \mathcal{O}_{\rho\lambda}^{T_2} + \eta_{\mu\rho} \mathcal{O}_{\nu\lambda}^{T_2} \right]
 \end{aligned} \tag{2.3}$$

where

$$\mathcal{O}_{\mu\nu}^{W_2} = \bar{\psi} \gamma_\mu D_\nu \psi$$

$$\begin{aligned}
 \mathcal{O}_{\mu\nu\sigma}^{W_3} &= \bar{\psi}\gamma_\mu D_\nu D_\sigma \psi \\
 \mathcal{O}_{\mu\nu\sigma}^{T_2} &= \bar{\psi}\sigma_{\mu\nu} D_\sigma \psi
 \end{aligned}
 \tag{2.4}$$

which are clearly consistent with the definitions in [21–23]. We note that $\mathcal{O}_S^{W_3}$ is the intermediate definition of the symmetrized operator. As in [21–23] we have derived the d -dimensional versions. Although the lattice computations are in strictly four dimensions, we will dimensionally regularize in $d = 4 - 2\epsilon$ dimensions where ϵ plays the role of the regularizing parameter. The renormalization constants will have a Laurent series in ϵ when subtracted in the $\overline{\text{MS}}$ scheme. The main calculational tool is the use of the MINCER algorithm derived in [32]. It is ideal for the present work as it evaluates massless 2-point Feynman diagrams to the finite part at three loops in dimensional regularization. The correlation function (2.1) clearly falls into this category and does not require infrared rearrangement or external momentum nullification. More practically the MINCER algorithm has been encoded in the powerful symbolic manipulation language FORM, [33], in [34], which therefore allows for a fully automated computation. For instance, once the Feynman rules for the operators are derived consistently then the algorithm is applied to produce the finite parts. Crucial to this is the electronic generation of the Feynman diagrams via the QGRAF package, [35]. These are then converted into FORM input notation for application of the MINCER algorithm by systematically including the Lorentz and colour indices for the gluon, quark and Faddeev-Popov ghost fields. For the present calculation there are 1 one loop, 8 two loop and 109 three loop Feynman graphs to be evaluated in principle for every combination of operators in (2.2) we consider here. These totals include graphs where there are two covariant derivatives in each operator as occurs for the W_3 sector. In the QGRAF generation of graphs we restrict the diagrams to the one particle irreducible, no tadpole and no snail set-up since we are dealing with massless fields. So, for example, there are no closed gluon loops at the location of an operator. Such graphs are trivially zero in dimensional regularization but may arise in, say, a lattice regularization. That aside, when, for example, $\partial\partial W_3$ is part of the correlator the majority of the 109 three loop graphs will in fact be trivially absent. For practical purposes it is best to have the most general set of diagrams for the full calculation rather than design QGRAF routines for specific cases and potentially omit graphs which contribute. By the same token the presence of the covariant derivatives in the W_3 sector means that the explicit evaluation is slowed significantly on the available computers. Therefore, when this occurred we chose to run each Lorentz projection individually in series, which improved run times substantially. Given this we note the final aspect of our notation and that is the decomposition of the correlation function into the explicit scalar amplitudes, $\Pi_{(k)}^{ij}(q)$. These are defined by

$$\Pi_{\mu_1\dots\mu_{n_i}\nu_1\dots\nu_{n_j}}^{ij}(q^2) = \sum_{k=1}^{n_{ij}} \mathcal{P}_{(k)\{\mu_1\dots\mu_{n_i}|\nu_1\dots\nu_{n_j}\}}^{ij}(q) \Pi_{(k)}^{ij}(q)
 \tag{2.5}$$

where $\mathcal{P}_{(k)\{\mu_1\dots\mu_{n_i}|\nu_1\dots\nu_{n_j}\}}^{ij}(q)$ are the Lorentz projectors. The subscript (k) (and also (l) later) label the sector. How these projectors are derived and their explicit forms are relegated to appendix A. However, we note the number of projectors for each of the eight

ij	S, S	V, V	T, T	V, W_2	V, W_3	W_2, W_2	W_3, W_3	T, T_2
n_{ij}	1	2	2	2	2	3	4	4

Table 1. Number of projectors for each correlation function sector.

correlation function sectors we focus on here, n_{ij} , is given in table 1. Clearly the number of projectors increases with the number of free Lorentz indices. Given this decomposition then to find each individual scalar amplitude $\Pi_{(k)}^{ij}(q)$ we multiply (2.5) by the appropriate element of the inverse projection tensor which is defined for each sector in appendix A. It is then this Lorentz scalar object which is put through the MINCER algorithm.

As is usual with a renormalizable quantum field theory each amplitude is divergent. To extract the explicit divergence we follow the algorithm of [36] derived from automatic multiloop computations. In general terms, one computes the Green's function as a function of bare parameters. Then the renormalized variables are introduced by the simple rescaling definition. For example, for the bare and renormalized coupling constants g_o and g respectively, we use $g_o = gZ_g$ where Z_g is the coupling constant renormalization constant. Though in dimensional regularization we will use $g_o = \mu^\epsilon gZ_g$ where μ is the arbitrary renormalization scale present due to the regularization. Here the explicit renormalization constants for the operator correlation functions are derived and discussed at length in the next section where the associated renormalization constants are also constructed. It suffices to say at this point that there is an extension to the algorithm of [36] in that the bare operators in the Green's function have to be rescaled to their renormalized operator without neglecting the mixing into other operators. Moreover, the explicit forms of the gauge independent operator correlation function renormalization constants are equally as important as the finite parts of the correlators from the point of view of assisting with the matching of lattice results to the corresponding continuum values in the high energy limit.

There is one concern with relation to the topology of the graphs which needs to be addressed. That is that we need to ensure that within our computer programmes we are in fact calculating the correlation functions of flavour non-singlet operators as opposed to flavour singlet ones. For instance, without the presence of a flavour matrix of some sort graphs which have a closed quark loop and only include one of \mathcal{O}^1 or \mathcal{O}^2 but not both together must be set to zero. If not it would be a contribution to a flavour singlet operator correlator. Therefore, whilst we have formally omitted flavour indices in the definition (2.1), our Feynman rules for the operators actually include a flavour matrix for each operator. Denoting this by λ^i where i labels the left operator 1 or right operator 2, then at an appropriate point of the computation terms with $\text{tr}(\lambda^1)\text{tr}(\lambda^2)$ are set to zero to only leave terms proportional to $\text{tr}(\lambda^1\lambda^2)$. This is then formally set to unity since it flags the flavour non-singlet contribution uniquely.

Finally, we comment on how we have chosen the correlation functions presented here. The set we consider is listed in table 1. First, our choice is motivated by ensuring that the corresponding lattice calculation has a minimal set of covariant derivatives to handle. Second, we are constrained by the masslessness of the problem. For instance, as is clear

from figure 1 and (2.1) we are dealing with closed quark loops. Therefore, one must have an even number of γ -matrices. As the quarks are massless and each quark propagator has exactly one γ -matrix the sum of γ -matrices present in both operators of (2.1) has to be even. Therefore, whilst from a lattice point of view it would be simple to have the quark mass operator as the off-diagonal element for the W_2 and W_3 sectors this correlator is trivially zero. In other words in the presence of quarks with generic mass m_q then the correlator will vanish as $O(m_q^\xi)$ where $\xi > 0$. Hence, for W_2 and W_3 they have to be paired with V . Similarly as T_2 involves $\sigma^{\mu\nu}$ one requires an even number of γ -matrices for the other operator of the correlator. Naively one would assume that this natural pairing would be with S . However, that leaves free Lorentz indices only on one operator and for T_2 , given the symmetry properties of the operator itself via $\sigma^{\mu\nu}$, it is not in fact possible to decompose the correlator into Lorentz tensors built from the metric tensor, $\eta^{\mu\nu}$, and the momentum, q^μ . Therefore, we have had to pair T_2 with T . Whilst these off-diagonal operators will probably be the ones of most interest to lattice computations we have chosen to consider the diagonal sectors $\{W_2, W_2\}$ and $\{W_3, W_3\}$ as well. There are several reasons for this. With a second avenue available to extract information on all the W_2 and W_3 sector renormalization constants used for the lattice, these will actually give useful consistency checks provided sufficient computation power is available for the lattice calculations. Next, the operator mixing issue is a novel feature of these correlators and we choose to consider them to ensure that we have obtained the correct overall picture of view from a calculational and renormalization point of view. A final, less firm, motivation is that the diagonal correlators of quark bilinear currents are useful to derive decay rates via the R -ratio formalism. (See, for example, [26–28].) Whilst those for W_2 and W_3 are tenuous in this respect, and we take them no further than finding the amplitudes, we do evaluate $\{T, T\}$ for this reason and construct the corresponding R -ratio for the tensor current as a by-product of our full computation.

3 Renormalization group

In this section we concentrate on general aspects of the renormalization of the operators we are interested in, (2.2), and the construction of the renormalization group equations satisfied by the renormalized operator correlation functions. There are essentially two parts to this. The first relates to the operator mixing which is a separate exercise unrelated to the correlation functions, whilst the second is an application of the mixing property.

For the operator mixing issue we note first that the quark current operators S , V and T are clearly multiplicatively renormalizable in the chiral limit and mass independent renormalization schemes. Therefore, we concentrate on the three sectors W_2 , W_3 and T_2 and justify our choice of operator basis. The first and last of these are similar, so we will specifically consider W_2 and W_3 . First, for W_2 we note that the usual Lorentz symmetric and traceless twist-2 flavour non-singlet operator used in deep inelastic scattering is

$$\mathcal{O}_{\mu\nu}^{W_2} = S\bar{\psi}\gamma_\mu D_\nu\psi . \tag{3.1}$$

This is not independent since one can of course add the independent operator $\mathcal{S} (D_\nu \bar{\psi}) \gamma_\mu \psi$ to the set of operators with the same symmetry properties. However, we want to make use of known renormalization results for $\mathcal{O}_{\mu\nu}^{W_2}$, [1–7, 37, 38], and using the latter noncanonical operator, whilst not difficult from a technical point of view, is not the only operator independent of it. Instead the operator

$$\mathcal{O}_{\mu\nu}^{\partial W_2} = \mathcal{S} \partial_\mu (\bar{\psi} \gamma_\nu \psi) \tag{3.2}$$

is independent of W_2 and with ∂W_2 and W_2 we can obtain $\mathcal{S} (D_\nu \bar{\psi}) \gamma_\mu \psi$ as a linear combination. Likewise for the T_2 sector the analogous basis is

$$\begin{aligned} \mathcal{O}_{\mu\nu\sigma}^{T_2} &= \mathcal{S} \bar{\psi} \sigma_{\mu\nu} D_\sigma \psi \\ \mathcal{O}_{\mu\nu\sigma}^{\partial T_2} &= \mathcal{S} \partial_\mu (\bar{\psi} \sigma_{\nu\sigma} \psi) . \end{aligned} \tag{3.3}$$

At the next sector higher one now has an object with three Lorentz indices and so one would expect three independent operators. Again we wish to retain the renormalization properties of the parent operator which means choosing the basis as $\{W_3, \partial W_3, \partial\partial W_3\}$ where

$$\begin{aligned} \mathcal{O}_{\mu\nu\sigma}^{W_3} &= \mathcal{S} \bar{\psi} \gamma_\mu D_\nu D_\sigma \psi \\ \mathcal{O}_{\mu\nu\sigma}^{\partial W_3} &= \mathcal{S} \partial_\mu (\bar{\psi} \gamma_\nu D_\sigma \psi) \\ \mathcal{O}_{\mu\nu\sigma}^{\partial\partial W_3} &= \mathcal{S} \partial_\mu \partial_\nu (\bar{\psi} \gamma_\sigma \psi) . \end{aligned} \tag{3.4}$$

Clearly the latter two are total derivatives of the set $\{W_2, \partial W_2\}$ and if one were to extend to the next sector level that set would involve the parent W_4 and the total derivatives of the W_3 sector. From the explicit calculation of the operator anomalous dimensions there is an important computational advantage from choosing the basis in this way. Though it should be stressed that at higher levels the choice of basis is arbitrary and one could in principle choose, say, $\mathcal{S} \partial_\mu ((D_\nu \bar{\psi}) \gamma_\sigma \psi)$ as an independent member of the set.

With the choices we have detailed for each sector, there is mixing under renormalization but clearly each mixing matrix of renormalization constants, $Z_{ij}^{\mathcal{O}}$, is upper triangular where

$$\mathcal{O}_{oi} = Z_{ij}^{\mathcal{O}} \mathcal{O}_j \tag{3.5}$$

relates bare operators, denoted by the subscript o , to their renormalized version. To be explicit the matrix for the W_2 and T_2 sectors is

$$Z_{ij}^{\mathcal{O}} = \begin{pmatrix} Z_{11}^{\mathcal{O}} & Z_{12}^{\mathcal{O}} \\ 0 & Z_{22}^{\mathcal{O}} \end{pmatrix} \tag{3.6}$$

and that for W_3 is

$$Z_{ij}^{\mathcal{O}} = \begin{pmatrix} Z_{11}^{\mathcal{O}} & Z_{12}^{\mathcal{O}} & Z_{13}^{\mathcal{O}} \\ 0 & Z_{22}^{\mathcal{O}} & Z_{23}^{\mathcal{O}} \\ 0 & 0 & Z_{33}^{\mathcal{O}} \end{pmatrix} . \tag{3.7}$$

Here we have chosen to simplify our notation by using numbers to denote the mixing matrix elements rather than the more clumsy $\{W_2, \partial W_2\}$, $\{W_3, \partial W_3, \partial\partial W_3\}$ or $\{T_2, \partial T_2\}$

as subscripts. Given these matrices we then define our anomalous dimension mixing matrix elements, $\gamma_{ij}^{\mathcal{O}}(a)$, formally as

$$\gamma_{ij}^{\mathcal{O}} = \mu \frac{d}{d\mu} \ln Z_{ij}^{\mathcal{O}} \tag{3.8}$$

where

$$\mu \frac{d}{d\mu} = \beta(a) \frac{\partial}{\partial a} + \alpha \gamma_{\alpha}(a, \alpha) \frac{\partial}{\partial \alpha}. \tag{3.9}$$

Here $\beta(a)$ is the β -function and $\gamma_{\alpha}(a, \alpha)$ is the anomalous dimension of the gauge parameter where we follow the conventions used in [21] to define its renormalization. Although all our renormalization constants will in fact be independent of α we have included it in (3.9) as it technically appears as a formal parameter in the QCD Lagrangian. For a renormalization where there is operator mixing, (3.8) is invariably given as the formal definition of the anomalous dimensions. However, for practical purposes in the derivation of the operator correlation function anomalous dimensions it is more appropriate to give the explicit consequences of (3.8). For sectors W_2 and T_2 we have

$$\begin{aligned} 0 &= \gamma_{11}^{\mathcal{O}}(a) Z_{11}^{\mathcal{O}} + \mu \frac{d}{d\mu} Z_{11}^{\mathcal{O}} \\ 0 &= \gamma_{11}^{\mathcal{O}}(a) Z_{12}^{\mathcal{O}} + \gamma_{12}^{\mathcal{O}}(a) Z_{22}^{\mathcal{O}} + \mu \frac{d}{d\mu} Z_{12}^{\mathcal{O}} \\ 0 &= \gamma_{22}^{\mathcal{O}}(a) Z_{22}^{\mathcal{O}} + \mu \frac{d}{d\mu} Z_{22}^{\mathcal{O}} \end{aligned} \tag{3.10}$$

and for W_3 we have similar relations,

$$\begin{aligned} 0 &= \gamma_{11}^{\mathcal{O}}(a) Z_{11}^{\mathcal{O}} + \mu \frac{d}{d\mu} Z_{11}^{\mathcal{O}} \\ 0 &= \gamma_{11}^{\mathcal{O}}(a) Z_{12}^{\mathcal{O}} + \gamma_{12}^{\mathcal{O}}(a) Z_{22}^{\mathcal{O}} + \mu \frac{d}{d\mu} Z_{12}^{\mathcal{O}} \\ 0 &= \gamma_{11}^{\mathcal{O}}(a) Z_{13}^{\mathcal{O}} + \gamma_{12}^{\mathcal{O}}(a) Z_{23}^{\mathcal{O}} + \gamma_{13}^{\mathcal{O}}(a) Z_{33}^{\mathcal{O}} + \mu \frac{d}{d\mu} Z_{13}^{\mathcal{O}} \\ 0 &= \gamma_{22}^{\mathcal{O}}(a) Z_{22}^{\mathcal{O}} + \mu \frac{d}{d\mu} Z_{22}^{\mathcal{O}} \\ 0 &= \gamma_{22}^{\mathcal{O}}(a) Z_{23}^{\mathcal{O}} + \gamma_{23}^{\mathcal{O}}(a) Z_{33}^{\mathcal{O}} + \mu \frac{d}{d\mu} Z_{23}^{\mathcal{O}} \\ 0 &= \gamma_{33}^{\mathcal{O}}(a) Z_{33}^{\mathcal{O}} + \mu \frac{d}{d\mu} Z_{33}^{\mathcal{O}}. \end{aligned} \tag{3.11}$$

We now turn to the practical problem of evaluating the anomalous dimensions explicitly to the loop order necessary for the operator correlation function renormalization at three loops. Whilst this is actually $O(a^2)$ we have determined the mixing matrices to $O(a^3)$ partly for completeness and checking reasons but also because of their potential use in phenomenological problems where a momentum flows out through the operator itself. For the mixing matrix anomalous dimensions the main issue is the determination of the off-diagonal elements. In the original approach of [1–3] the operators were inserted in the Green’s function $\langle \psi(p) \mathcal{O}(0) \bar{\psi}(-p) \rangle$ whose more general version is illustrated graphically in figure 2. As

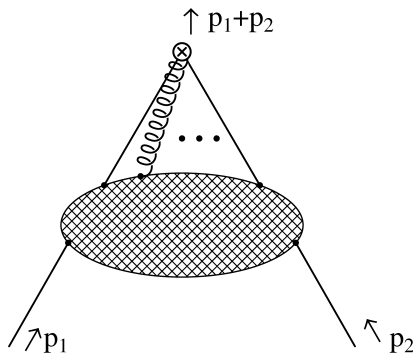


Figure 2. Green’s function, $\langle \psi(p_1) \mathcal{O}^i(-p_1 - p_2) \bar{\psi}(p_2) \rangle$, used to renormalize the operators \mathcal{O}^i .

the operator is a zero momentum insertion the contribution to the renormalization of the off-diagonal part of $Z_{ij}^{\mathcal{O}}$ cannot be deduced as the total derivative operator insertions vanish trivially for this momentum configuration. So only all the diagonal elements of our mixing matrix can be deduced via this momentum routing. There are two remaining choices for routing momenta with one nullification, which is necessary if we wish to apply MINCER as our tool of computation. With the choice $\langle \psi(p) \mathcal{O}(-p) \bar{\psi}(0) \rangle$ it is clear that one can access the elements $Z_{12}^{W_2}$, $Z_{12}^{T_2}$ and $Z_{23}^{W_3}$ once the respective values for $Z_{11}^{W_2}$, $Z_{11}^{T_2}$ and $Z_{22}^{W_3}$ have been included from the original results of [1–7, 39–41]. The determination of $Z_{12}^{W_3}$ and $Z_{13}^{W_3}$ is more difficult. This is because there is only *one* momentum choice which makes contact with both these renormalization constants at the same time. We circumvented this difficulty to two loops by not omitting the terms involving $\ln(p^2/\mu^2)$ which derive from the dimensionality of the loop integrals at each loop order. One ordinarily ignores such terms in a renormalization since they cancel trivially in a renormalizable theory. Retaining them means that divergences such as $\frac{1}{\epsilon} \ln(p^2/\mu^2)$, $\frac{1}{\epsilon^2} \ln(p^2/\mu^2)$ and $\frac{1}{\epsilon} (\ln(p^2/\mu^2))^2$ are present at various loop orders but some of their coefficients involve counterterm parts of $Z_{12}^{W_3}$ and $Z_{13}^{W_3}$ in their coupling constant series and Laurent expansion in ϵ . They give extra constraints on $Z_{12}^{W_3}$ and $Z_{13}^{W_3}$ which allowed us to decipher the W_3 sector mixing matrix to *two* loops. At three loops we can only derive a simple relation between the simple poles of each renormalization constant. However, this is not in fact necessary for our operator correlator results. Again the main tool is the MINCER package written in FORM, [35–37]. The graphs are again generated by QGRAF, [38]. For the maximum number of possible covariant derivative terms that can arise, there are 3 one loop, 37 two loop and 684 three loop Feynman graphs to evaluate. As before these are one particle irreducible graphs without snails or tadpoles.

Given these considerations it is evident that sector W_4 would be more difficult to deduce completely to three loops due to a similar momentum routing issue. However, it could be determined to two loops in principle by performing a four loop calculation with $\ln(p^2/\mu^2)$ terms included. If there were a four loop MINCER routine available then this would be a viable proposition. Therefore, in principle, the procedure to determine the mixing matrix for the non-singlet sectors W_n is available but in practice requires the

computational machinery to evaluate the off-diagonal mixing matrix elements. All that remains is to record the explicit values which are all given in the $\overline{\text{MS}}$ scheme. First, for completeness and for comparing with the conventions of previous calculations the vector and tensor current anomalous dimensions, [39–41], are

$$\begin{aligned}
 \gamma^V(a) &= O(a^4) \\
 \gamma^T(a) &= C_F a + [257C_A - 171C_F - 52T_F N_f] \frac{C_F a^2}{18} \\
 &\quad + [13639C_A^2 - 4320\zeta(3)C_A^2 + 12096\zeta(3)C_A C_F \\
 &\quad - 20469C_A C_F - 1728\zeta(3)C_A T_F N_f - 4016C_A T_F N_f \\
 &\quad - 6912\zeta(3)C_F^2 + 6570C_F^2 + 1728\zeta(3)C_F T_F N_f \\
 &\quad + 1176C_F T_F N_f - 144T_F^2 C_F^2] \frac{C_F a^3}{108} + O(a^4) \tag{3.12}
 \end{aligned}$$

where $\zeta(z)$ is the Riemann zeta function, C_A and C_F are the usual colour group Casimirs and T_F is defined by $\text{Tr}(T^a T^b) = T_F \delta^{ab}$ where T^a are the generators of the colour group. For the two sectors with two operators we have

$$\begin{aligned}
 \gamma_{11}^{W_2}(a) &= \frac{8}{3} C_F a + \frac{1}{27} [376C_A C_F - 112C_F^2 - 128C_F T_F N_f] a^2 \\
 &\quad + \frac{1}{243} [(5184\zeta(3) + 20920) C_A^2 C_F - (15552\zeta(3) + 8528) C_A C_F^2 \\
 &\quad - (10368\zeta(3) + 6256) C_A C_F T_F N_f + (10368\zeta(3) - 560) C_F^3 \\
 &\quad + (10368\zeta(3) - 6824) C_F^2 T_F N_f - 896C_F T_F^2 N_f^2] a^3 + O(a^4) \\
 \gamma_{12}^{W_2}(a) &= -\frac{4}{3} C_F a + \frac{1}{27} [56C_F^2 - 188C_A C_F + 64C_F T_F N_f] a^2 \\
 &\quad + \frac{1}{243} [(7776\zeta(3) + 4264) C_A C_F^2 - (2592\zeta(3) + 10460) C_A^2 C_F \\
 &\quad + (5184\zeta(3) + 3128) C_A C_F T_F N_f - (5184\zeta(3) - 280) C_F^3 \\
 &\quad - (5184\zeta(3) - 3412) C_F^2 T_F N_f + 448C_F T_F^2 N_f^2] a^3 + O(a^4) \\
 \gamma_{22}^{W_2}(a) &= O(a^4) \tag{3.13}
 \end{aligned}$$

and

$$\begin{aligned}
 \gamma_{11}^{T_2}(a) &= 3C_F a + \frac{1}{2} [35C_A C_F - 9C_F^2 - 12C_F T_F N_f] a^2 \\
 &\quad + \frac{1}{108} [12553C_A^2 C_F - 7479C_A C_F^2 + 1782C_F^3 - (5184\zeta(3) + 4168) C_A C_F T_F N_f \\
 &\quad + (5184\zeta(3) - 3240) C_F^2 T_F N_f - 368C_F T_F^2 N_f^2] a^3 + O(a^4) \\
 \gamma_{12}^{T_2}(a) &= -C_F a + \frac{1}{18} [28C_F T_F N_f - 45C_F^2 - 29C_A C_F] a^2 \\
 &\quad + \frac{1}{108} [(6048\zeta(3) - 6495) C_A C_F^2 - (2160\zeta(3) - 543) C_A^2 C_F \\
 &\quad + (1728\zeta(3) + 76) C_A C_F T_F N_f - (3456\zeta(3) - 2394) C_F^3 \\
 &\quad - (1728\zeta(3) - 2208) C_F^2 T_F N_f + 112C_F T_F^2 N_f^2] a^3 + O(a^4)
 \end{aligned}$$

$$\begin{aligned}
 \gamma_{22}^{T_2}(a) &= C_F a + \frac{1}{18} [257C_A C_F - 171C_F^2 - 52C_F T_F N_f] a^2 \\
 &+ \frac{1}{108} \left[(13639 - 4320\zeta(3)) C_A^2 C_F + (12096\zeta(3) - 20469) C_A C_F^2 \right. \\
 &- (1728\zeta(3) + 4016) C_A C_F T_F N_f - (6912\zeta(3) - 6570) C_F^3 \\
 &\left. + (1728\zeta(3) + 1176) C_F^2 T_F N_f - 144C_F T_F^2 N_f^2 \right] a^3 + O(a^4). \tag{3.14}
 \end{aligned}$$

Given the issues with computing the full set of anomalous dimensions to three loops for the W_3 sector we record that the results are

$$\begin{aligned}
 \gamma_{11}^{W_3}(a) &= \frac{25}{6} C_F a + \frac{1}{432} [8560C_A C_F - 2035C_F^2 - 3320C_F T_F N_f] a^2 \\
 &+ \frac{1}{15552} \left[(285120\zeta(3) + 1778866) C_A^2 C_F - (855360\zeta(3) + 311213) C_A C_F^2 \right. \\
 &- (1036800\zeta(3) + 497992) C_A C_F T_F N_f + (570240\zeta(3) - 244505) C_F^3 \\
 &\left. + (1036800\zeta(3) - 814508) C_F^2 T_F N_f - 82208C_F T_F^2 N_f^2 \right] a^3 + O(a^4) \\
 \gamma_{12}^{W_3}(a) &= -\frac{3}{2} C_F a + \frac{1}{144} [81C_F^2 - 848C_A C_F + 424C_F T_F N_f] a^2 + O(a^3) \\
 \gamma_{13}^{W_3}(a) &= -\frac{1}{2} C_F a + \frac{1}{144} [103C_F^2 - 388C_A C_F + 104C_F T_F N_f] a^2 + O(a^3) \\
 \gamma_{22}^{W_3}(a) &= \frac{8}{3} C_F a + \frac{1}{27} [376C_A C_F - 112C_F^2 - 128C_F T_F N_f] a^2 \\
 &+ \frac{1}{243} \left[(5184\zeta(3) + 20920) C_A^2 C_F - (15552\zeta(3) + 8528) C_A C_F^2 \right. \\
 &- (10368\zeta(3) + 6256) C_A C_F T_F N_f + (10368\zeta(3) - 560) C_F^3 \\
 &\left. + (10368\zeta(3) - 6824) C_F^2 T_F N_f - 896C_F T_F^2 N_f^2 \right] a^3 + O(a^4) \\
 \gamma_{23}^{W_3}(a) &= -\frac{4}{3} C_F a + \frac{1}{27} [56C_F^2 - 188C_A C_F + 64C_F T_F N_f] a^2 \\
 &+ \frac{1}{243} \left[(7776\zeta(3) + 4264) C_A C_F^2 - (2592\zeta(3) + 10460) C_A^2 C_F \right. \\
 &+ (5184\zeta(3) + 3128) C_A C_F T_F N_f - (5184\zeta(3) - 280) C_F^3 \\
 &\left. - (5184\zeta(3) - 3412) C_F^2 T_F N_f + 448C_F T_F^2 N_f^2 \right] a^3 + O(a^4) \\
 \gamma_{33}^{W_3}(a) &= O(a^4) \tag{3.15}
 \end{aligned}$$

where those for $\gamma_{12}^{W_3}(a)$ and $\gamma_{13}^{W_3}(a)$ are only given to two loops. Clearly the diagonal anomalous dimensions of each sector, including those with total derivatives, are the same as the corresponding non-total derivative operator. Equally the mixing of the second row of W_3 is trivially related to that of the first row of W_2 . This is a reassuring observation. As further checks on the results, since we have used the algorithm of [36] to determine the renormalization constants, therefore, the double and triple poles in ϵ are predetermined by the renormalization group equations of (3.10) and (3.11). We note that our results are consistent with those constraints.

We now focus on the renormalization structure of the operator correlation functions. These are Green's functions of operators rather than of fields but like Green's functions of

fields they have an associated renormalization constant after the constituent bare operators have been replaced by their renormalized versions, taking into account any mixing. We denote these additional renormalization constants by $Z_{(k)}^{ij}$ and follow the quark current correlator renormalization formalism of [26–28]. This renormalization constant appears as a contact term rather than as a canonical multiplicative renormalization constant that one normally expects in the renormalization of a Green’s function involving only fields. In the case of operator correlators where there is no mixing of the constituent operators, the relation between bare and renormalized correlators has been given in, for example, [26–28]. For completeness, we give the form for the tensor current correlation function. It is

$$\Pi_{(i)}^{T,T}(q) = Z_{(i)}^{T,T} q^2 + \mu^{2\epsilon} \left(Z^T \right)^2 \Pi_o^{T,T}(q) \quad (3.16)$$

where we have included the subscript label deriving from the Lorentz tensor decomposition since there will in principle be a divergence for each projection. This relation is the basic form used in our computer algebra setup and the automatic Feynman diagram renormalization procedure of [36] is easy to extend and encode in FORM for this case. Given (3.16) it is straightforward to derive the renormalization group equation satisfied by the renormalized correlation function. Applying (3.9) to (3.16) we have

$$0 = \mu \frac{d}{d\mu} \Pi_{(i)}^{T,T}(q) + 2\gamma^T(a) \Pi_{(i)}^{T,T}(q) - q^2 \gamma_{(i)}^{T,T}(a) \quad (3.17)$$

where the correlation function anomalous dimension is formally given by

$$\gamma_{(i)}^{T,T}(a) = \left[-\epsilon + \beta(a) \frac{\partial}{\partial a} + 2\gamma^T(a) \right] Z_{(i)}^{T,T}. \quad (3.18)$$

Although part of our ultimate aim is to provide the finite parts of the amplitudes we will also determine these anomalous dimensions to $O(a^2)$ inclusive. Indeed we use the results, such as (3.18) as consistency checks on the explicit renormalization constants. Aside from being gauge independent expressions, the double and triple poles in ϵ are again determined by the lower order simple poles. Moreover, if one had not taken the issue of mixing into account this internal consistency check would in fact fail.

Given this form for the quark current correlators, S and V , [26–28], and now also T , their extension to operator correlators where there is mixing is subtle. Therefore, we highlight the structure for two cases which are $\{V, W_3\}$ and $\{W_3, W_3\}$. The resulting renormalization group functions for the remaining cases, $\{V, W_2\}$, $\{W_2, W_2\}$ and $\{T, T_2\}$ can be readily deduced from the final relations by appropriate relabelling and ignoring irrelevant terms which, say, do not occur for the W_2 sector. For each case we consider, the key is to simply write down the forms analogous to (3.16) but including all possible consistent mixings and all possible operator combinations in the correlation functions for that sector and ensure the equation is dimensionally consistent. The reason for this is that the relation between bare and renormalized correlation functions are entwined in an intricate way. As the $\{V, W_3\}$ case is simple since only one of the constituent operator undergoes mixing, we

illustrate this by giving the three renormalization definitions explicitly as,

$$\begin{aligned}
 \Pi_{(i)}^{V,W_3}(q) &= Z_{(i)}^{V,W_3}(q^2)^2 + \mu^{2\epsilon} Z^V \left[Z_{11}^{W_3} \Pi_{o(i)}^{V,W_3}(q) + Z_{12}^{W_3} \Pi_{o(i)}^{V,\partial W_3}(q) + Z_{13}^{W_3} \Pi_{o(i)}^{V,\partial\partial W_3}(q) \right] \\
 \Pi_{(i)}^{V,\partial W_3}(q) &= Z_{(i)}^{V,\partial W_3}(q^2)^2 + \mu^{2\epsilon} Z^V \left[Z_{22}^{W_3} \Pi_{o(i)}^{V,\partial W_3}(q) + Z_{23}^{W_3} \Pi_{o(i)}^{V,\partial\partial W_3}(q) \right] \\
 \Pi_{(i)}^{V,\partial\partial W_3}(q) &= Z_{(i)}^{V,\partial\partial W_3}(q^2)^2 + \mu^{2\epsilon} Z^V Z_{33}^{W_3} \Pi_{o(i)}^{V,\partial\partial W_3}(q)
 \end{aligned} \tag{3.19}$$

where the mixing matrix elements of (3.7) appear and the factor of q^2 multiplying $Z_{(k)}^{i,j}$ derives from the dimensionality of the actual correlator in question to ensure a dimensionless renormalization constant. The next stage is to apply (3.9) to each of equation and then rewrite the full set without any bare correlators. This is algebraically tedious but it is best to start with the final equation since it is similar to (3.16) whence

$$0 = \mu \frac{d}{d\mu} \Pi_{(i)}^{V,\partial\partial W_3}(q) + \left(\gamma^V(a) + \gamma_{33}^{W_3}(a) \right) \Pi_{(i)}^{V,\partial\partial W_3}(q) - (q^2)^2 \gamma_{(i)}^{V,\partial\partial W_3}(a) \tag{3.20}$$

with

$$\gamma_{(i)}^{V,\partial\partial W_3}(a) = \left[-\epsilon + \beta(a) \frac{\partial}{\partial a} + \gamma^V(a) + \gamma_{33}^{W_3}(a) \right] Z_{(i)}^{V,\partial\partial W_3}. \tag{3.21}$$

Considering the second equation of (3.19) next, after the application of (3.9) it is necessary to rewrite both bare correlators which now occur back in terms of their renormalized versions. Throughout this and all our other similar manipulations, we always associate terms with powers of the momentum q^2 as contributing to the correlator anomalous dimension. Hence, we have

$$\begin{aligned}
 0 &= \mu \frac{d}{d\mu} \Pi_{(i)}^{V,\partial W_3}(q) + \left(\gamma^V(a) + \gamma_{22}^{W_3}(a) \right) \Pi_{(i)}^{V,\partial W_3}(q) \\
 &\quad + \gamma_{23}^{W_3}(a) \Pi_{(i)}^{V,\partial\partial W_3}(q) - (q^2)^2 \gamma_{(i)}^{V,\partial W_3}(a)
 \end{aligned} \tag{3.22}$$

and

$$\gamma_{(i)}^{V,\partial W_3}(a) = \left[-\epsilon + \beta(a) \frac{\partial}{\partial a} + \gamma^V(a) + \gamma_{22}^{W_3}(a) \right] Z_{(i)}^{V,\partial W_3} + \gamma_{23}^{W_3}(a) Z_{(i)}^{V,\partial\partial W_3}. \tag{3.23}$$

Finally, both of the last equations of (3.19) are required to complete the set of three renormalization group functions for the $\{V, W_3\}$ case. We have

$$\begin{aligned}
 0 &= \mu \frac{d}{d\mu} \Pi_{(i)}^{V,W_3}(q) + \left(\gamma^V(a) + \gamma_{11}^{W_3}(a) \right) \Pi_{(i)}^{V,W_3}(q) + \gamma_{12}^{W_3}(a) \Pi_{(i)}^{V,\partial W_3}(q) \\
 &\quad + \gamma_{13}^{W_3}(a) \Pi_{(i)}^{V,\partial\partial W_3}(q) - (q^2)^2 \gamma_{(i)}^{V,W_3}(a)
 \end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
 \gamma_{(i)}^{V,W_3}(a) &= \left[-\epsilon + \beta(a) \frac{\partial}{\partial a} + \gamma^V(a) + \gamma_{11}^{W_3}(a) \right] Z_{(i)}^{V,W_3} \\
 &\quad + \gamma_{12}^{W_3}(a) Z_{(i)}^{V,\partial W_3} + \gamma_{13}^{W_3}(a) Z_{(i)}^{V,\partial\partial W_3}.
 \end{aligned} \tag{3.25}$$

For the $\{W_3, W_3\}$ case the derivation follows parallel lines though the starting point is a more complicated set of six relations between bare and renormalized correlators as there is mixing in both inserted operators. These are

$$\begin{aligned}
 \Pi_{(i)}^{W_3, W_3}(q) &= Z_{(i)}^{W_3, W_3}(q^2)^3 \\
 &+ \mu^{2\epsilon} \left[\left(Z_{11}^{W_3} \right)^2 \Pi_{o(i)}^{W_3, W_3}(q) + 2Z_{11}^{W_3} Z_{12}^{W_3} \Pi_{o(i)}^{W_3, \partial W_3}(q) \right. \\
 &+ 2Z_{11}^{W_3} Z_{13}^{W_3} \Pi_{o(i)}^{W_3, \partial \partial W_3}(q) + \left(Z_{12}^{W_3} \right)^2 \Pi_{o(i)}^{\partial W_3, \partial W_3}(q) \\
 &\left. + 2Z_{12}^{W_3} Z_{13}^{W_3} \Pi_{o(i)}^{\partial W_3, \partial \partial W_3}(q) + \left(Z_{13}^{W_3} \right)^2 \Pi_{o(i)}^{\partial \partial W_3, \partial \partial W_3}(q) \right] \\
 \Pi_{(i)}^{W_3, \partial W_3}(q) &= Z_{(i)}^{W_3, \partial W_3}(q^2)^3 \\
 &+ \mu^{2\epsilon} \left[Z_{11}^{W_3} Z_{22}^{W_3} \Pi_{o(i)}^{W_3, \partial W_3}(q) + Z_{12}^{W_3} Z_{22}^{W_3} \Pi_{o(i)}^{\partial W_3, \partial W_3}(q) \right. \\
 &+ Z_{13}^{W_3} Z_{22}^{W_3} \Pi_{o(i)}^{\partial W_3, \partial \partial W_3}(q) + Z_{11}^{W_3} Z_{23}^{W_3} \Pi_{o(i)}^{W_3, \partial \partial W_3}(q) \\
 &\left. + Z_{12}^{W_3} Z_{23}^{W_3} \Pi_{o(i)}^{\partial W_3, \partial \partial W_3}(q) + Z_{13}^{W_3} Z_{23}^{W_3} \Pi_{o(i)}^{\partial \partial W_3, \partial \partial W_3}(q) \right] \\
 \Pi_{(i)}^{W_3, \partial \partial W_3}(q) &= Z_{(i)}^{W_3, \partial \partial W_3}(q^2)^3 \\
 &+ \mu^{2\epsilon} \left[Z_{11}^{W_3} Z_{33}^{W_3} \Pi_{o(i)}^{W_3, \partial \partial W_3}(q) + Z_{12}^{W_3} Z_{33}^{W_3} \Pi_{o(i)}^{\partial W_3, \partial \partial W_3}(q) \right. \\
 &\left. + Z_{13}^{W_3} Z_{33}^{W_3} \Pi_{o(i)}^{\partial \partial W_3, \partial \partial W_3}(q) \right] \\
 \Pi_{(i)}^{\partial W_3, \partial W_3}(q) &= Z_{(i)}^{\partial W_3, \partial W_3}(q^2)^3 \\
 &+ \mu^{2\epsilon} \left[\left(Z_{22}^{W_3} \right)^2 \Pi_{o(i)}^{\partial W_3, \partial W_3}(q) + 2Z_{22}^{W_3} Z_{23}^{W_3} \Pi_{o(i)}^{\partial W_3, \partial \partial W_3}(q) \right. \\
 &\left. + \left(Z_{23}^{W_3} \right)^2 \Pi_{o(i)}^{\partial \partial W_3, \partial \partial W_3}(q) \right] \\
 \Pi_{(i)}^{\partial \partial W_3, \partial \partial W_3}(q) &= Z_{(i)}^{\partial \partial W_3, \partial \partial W_3}(q^2)^3 \\
 &+ \mu^{2\epsilon} \left[Z_{22}^{W_3} Z_{33}^{W_3} \Pi_{o(i)}^{\partial W_3, \partial \partial W_3}(q) + Z_{23}^{W_3} Z_{33}^{W_3} \Pi_{o(i)}^{\partial \partial W_3, \partial \partial W_3}(q) \right] \\
 \Pi_{(i)}^{\partial \partial W_3, \partial \partial W_3}(q) &= Z_{(i)}^{\partial \partial W_3, \partial \partial W_3}(q^2)^3 + \mu^{2\epsilon} \left(Z_{33}^{W_3} \right)^2 \Pi_{o(i)}^{\partial \partial W_3, \partial \partial W_3}(q). \tag{3.26}
 \end{aligned}$$

It is worth noting that our choice of the upper triangular form for the mixing matrix in fact leads to a simpler renormalization group equation derivation from the point of view of disentangling the relations to produce equations without bare correlators. Again for this sector it is best to begin deriving the full renormalization group equations from the final equation of (3.26) and then systematically move to the row immediately above in the matrix. As this exercise is equally as straightforward though more tedious than the $\{V, W_3\}$ case, we merely record that the final renormalization group equations are

$$\begin{aligned}
 0 &= \mu \frac{d}{d\mu} \Pi_{(i)}^{W_3, W_3}(q) + 2\gamma_{11}^{W_3}(a) \Pi_{(i)}^{W_3, W_3}(q) + 2\gamma_{12}^{W_3}(a) \Pi_{(i)}^{W_3, \partial W_3}(q) \\
 &+ 2\gamma_{13}^{W_3}(a) \Pi_{(i)}^{W_3, \partial \partial W_3}(q) - (q^2)^3 \gamma_{(i)}^{W_3, W_3}(a)
 \end{aligned}$$

$$\begin{aligned}
 0 &= \mu \frac{d}{d\mu} \Pi_{(i)}^{W_3, \partial W_3}(q) + \left(\gamma_{11}^{W_3}(a) + \gamma_{22}^{W_3}(a) \right) \Pi_{(i)}^{W_3, \partial W_3}(q) + \gamma_{12}^{W_3}(a) \Pi_{(i)}^{\partial W_3, \partial W_3}(q) \\
 &\quad + \gamma_{23}^{W_3}(a) \Pi_{(i)}^{\partial W_3, \partial \partial W_3}(q) + \gamma_{13}^{W_3}(a) \Pi_{(i)}^{\partial \partial W_3, \partial \partial W_3}(q) - (q^2)^3 \gamma_{(i)}^{W_3, \partial W_3}(a) \\
 0 &= \mu \frac{d}{d\mu} \Pi_{(i)}^{W_3, \partial \partial W_3}(q) + \left(\gamma_{11}^{W_3}(a) + \gamma_{33}^{W_3}(a) \right) \Pi_{(i)}^{W_3, \partial \partial W_3}(q) + \gamma_{12}^{W_3}(a) \Pi_{(i)}^{\partial W_3, \partial \partial W_3}(q) \\
 &\quad + \gamma_{13}^{W_3}(a) \Pi_{(i)}^{\partial \partial W_3, \partial \partial W_3}(q) - (q^2)^3 \gamma_{(i)}^{W_3, \partial \partial W_3}(a) \\
 0 &= \mu \frac{d}{d\mu} \Pi_{(i)}^{\partial W_3, \partial W_3}(q) + 2\gamma_{22}^{W_3}(a) \Pi_{(i)}^{\partial W_3, \partial W_3}(q) + 2\gamma_{23}^{W_3}(a) \Pi_{(i)}^{\partial W_3, \partial \partial W_3}(q) \\
 &\quad - (q^2)^3 \gamma_{(i)}^{\partial W_3, \partial W_3}(a) \\
 0 &= \mu \frac{d}{d\mu} \Pi_{(i)}^{\partial W_3, \partial \partial W_3}(q) + \left(\gamma_{22}^{W_3}(a) + \gamma_{33}^{W_3}(a) \right) \Pi_{(i)}^{\partial W_3, \partial \partial W_3}(q) + \gamma_{23}^{W_3}(a) \Pi_{(i)}^{\partial \partial W_3, \partial \partial W_3}(q) \\
 &\quad - (q^2)^3 \gamma_{(i)}^{\partial W_3, \partial \partial W_3}(a) \\
 0 &= \mu \frac{d}{d\mu} \Pi_{(i)}^{\partial \partial W_3, \partial \partial W_3}(q) + 2\gamma_{33}^{W_3}(a) \Pi_{(i)}^{\partial \partial W_3, \partial \partial W_3}(q) - (q^2)^3 \gamma_{(i)}^{\partial \partial W_3, \partial \partial W_3}(a) \tag{3.27}
 \end{aligned}$$

where the correlator anomalous dimensions are

$$\begin{aligned}
 \gamma_{(i)}^{W_3, W_3}(a) &= \left[-\epsilon + \beta(a) \frac{\partial}{\partial a} + 2\gamma_{11}^{W_3}(a) \right] Z_{(i)}^{W_3, W_3} \\
 &\quad + 2\gamma_{12}^{W_3}(a) Z_{(i)}^{W_3, \partial W_3} + 2\gamma_{13}^{W_3}(a) Z_{(i)}^{W_3, \partial \partial W_3} \\
 \gamma_{(i)}^{W_3, \partial W_3}(a) &= \left[-\epsilon + \beta(a) \frac{\partial}{\partial a} + \gamma_{11}^{W_3}(a) + \gamma_{22}^{W_3}(a) \right] Z_{(i)}^{W_3, \partial W_3} \\
 &\quad + \gamma_{12}^{W_3}(a) Z_{(i)}^{\partial W_3, \partial W_3} + \gamma_{23}^{W_3}(a) Z_{(i)}^{W_3, \partial \partial W_3} + \gamma_{13}^{W_3}(a) Z_{(i)}^{\partial W_3, \partial \partial W_3} \\
 \gamma_{(i)}^{W_3, \partial \partial W_3}(a) &= \left[-\epsilon + \beta(a) \frac{\partial}{\partial a} + \gamma_{11}^{W_3}(a) + \gamma_{33}^{W_3}(a) \right] Z_{(i)}^{W_3, \partial \partial W_3} \\
 &\quad + \gamma_{12}^{W_3}(a) Z_{(i)}^{\partial W_3, \partial \partial W_3} + \gamma_{13}^{W_3}(a) Z_{(i)}^{\partial \partial W_3, \partial \partial W_3} \\
 \gamma_{(i)}^{\partial W_3, \partial W_3}(a) &= \left[-\epsilon + \beta(a) \frac{\partial}{\partial a} + 2\gamma_{22}^{W_3}(a) \right] Z_{(i)}^{\partial W_3, \partial W_3} + 2\gamma_{23}^{W_3}(a) Z_{(i)}^{\partial W_3, \partial \partial W_3} \\
 \gamma_{(i)}^{\partial W_3, \partial \partial W_3}(a) &= \left[-\epsilon + \beta(a) \frac{\partial}{\partial a} + \gamma_{22}^{W_3}(a) + \gamma_{33}^{W_3}(a) \right] Z_{(i)}^{\partial W_3, \partial \partial W_3} + \gamma_{23}^{W_3}(a) Z_{(i)}^{\partial \partial W_3, \partial \partial W_3} \\
 \gamma_{(i)}^{\partial \partial W_3, \partial \partial W_3}(a) &= \left[-\epsilon + \beta(a) \frac{\partial}{\partial a} + 2\gamma_{33}^{W_3}(a) \right] Z_{(i)}^{\partial \partial W_3, \partial \partial W_3}. \tag{3.28}
 \end{aligned}$$

Comparing these final forms with the original relationships (3.26) an evident pattern emerges in the final renormalization group equations. Not all the original bare operators of (3.26) appear in the corresponding equation. However, this is partly because the transformation to (3.27) involves the non-trivial entanglement alluded to earlier but in such a way that no information is lost. In practice there are cancellations in the derivation in such a way that the coefficient of certain off-diagonal elements is zero. Indeed given the upper triangular form of the mixing matrix and the final forms (3.27), one could have been tempted merely to write these down without derivation.

For completeness, we close this section by recording the formal definitions of the remaining operator correlation function anomalous dimensions. As is apparent from com-

paring with their $\{V, W_3\}$ and $\{W_3, W_3\}$ counterparts there is a consistent correspondence between the terms of the anomalous dimensions and the form of the renormalization group function itself that means we only record the anomalous dimensions themselves for brevity and as an aid to checking the renormalization group equations. We have for those cases involving W_2

$$\begin{aligned}\gamma^{V,W_2}(a) &= \left[-\epsilon + \beta(a)\frac{\partial}{\partial a} + \gamma^V(a) + \gamma_{11}^{W_2}(a) \right] Z^{V,W_2} + \gamma_{12}^{W_2}(a) Z^{V,\partial W_2} \\ \gamma^{V,\partial W_2}(a) &= \left[-\epsilon + \beta(a)\frac{\partial}{\partial a} + \gamma^V(a) + \gamma_{22}^{W_2}(a) \right] Z^{V,\partial W_2}\end{aligned}\tag{3.29}$$

and

$$\begin{aligned}\gamma_{(i)}^{W_2,W_2}(a) &= \left[-\epsilon + \beta(a)\frac{\partial}{\partial a} + 2\gamma_{11}^{W_2}(a) \right] Z_{(i)}^{W_2,W_2} + 2\gamma_{12}^{W_2}(a) Z_{(i)}^{W_2,\partial W_2} \\ \gamma_{(i)}^{W_2,\partial W_2}(a) &= \left[-\epsilon + \beta(a)\frac{\partial}{\partial a} + \gamma_{11}^{W_2}(a) + \gamma_{22}^{W_2}(a) \right] Z_{(i)}^{W_2,\partial W_2} + \gamma_{12}^{W_2}(a) Z_{(i)}^{\partial W_2,\partial W_2} \\ \gamma_{(i)}^{\partial W_2,\partial W_2}(a) &= \left[-\epsilon + \beta(a)\frac{\partial}{\partial a} + 2\gamma_{22}^{W_2}(a) \right] Z_{(i)}^{\partial W_2,\partial W_2}.\end{aligned}\tag{3.30}$$

Finally, for T_2 we have

$$\begin{aligned}\gamma_{(i)}^{T,T_2}(a) &= \left[-\epsilon + \beta(a)\frac{\partial}{\partial a} + \gamma^T(a) + \gamma_{11}^{T_2}(a) \right] Z_{(i)}^{T,T_2} + \gamma_{12}^{T_2}(a) Z_{(i)}^{T,\partial T_2} \\ \gamma_{(i)}^{T,\partial T_2}(a) &= \left[-\epsilon + \beta(a)\frac{\partial}{\partial a} + \gamma^T(a) + \gamma_{22}^{T_2}(a) \right] Z_{(i)}^{T,\partial T_2}.\end{aligned}\tag{3.31}$$

4 Results

We now turn to the mundane task of recording all our results for the operator correlation functions. These are broken into subsections where the first named operator of the title corresponds to the operator \mathcal{O}^1 of figure 1. In each section, we provide the finite renormalized amplitudes with respect to the various projections and then the associated correlator anomalous dimension. In recording the finite parts of all our correlators we show the overall dimension of the amplitude by explicitly factorizing off the overall q^2 dependence which is not the same for each sector. We note that we included the powers of q^2 in the contact term of the relation between bare and renormalized amplitudes in the basic relation in order to identify the operator correlator anomalous dimensions in analysing the renormalization group structure. For certain sectors due to the nature of the total derivative operators the explicit form of some amplitudes appear in an earlier subsection since there is a clear relation to $O(a^2)$ with the explicit value of the amplitudes. In certain cases, such as the vector correlator, the vanishing of the correlator projection to three loops is actually an all orders feature due to symmetry. In other cases where relations hold to $O(a^2)$ this may be valid to all orders but we make no assertion beyond the order we have calculated to. Finally, for completeness as well as for comparing conventions, we also display results for

$\{S, S\}$ and $\{V, V\}$ which are in agreement with, [26–28]. With $d(R)$ the dimension of the quark representation and

$$\ell = \ln \left(\frac{\mu^2}{q^2} \right) \quad (4.1)$$

we have:

4.1 Scalar-scalar

$$\Pi^{S,S}(q) = q^2 \tilde{\Pi}^{S,S}(a) \quad (4.2)$$

$$\begin{aligned} \tilde{\Pi}^{S,S}(a) = d(R) & \left[4 + 2\ell + C_F \left[\frac{131}{2} - 24\zeta(3) + 34\ell + 6\ell^2 \right] a \right. \\ & + C_F \left[\left(64\zeta(3) - \frac{2044}{9} - 130\ell + 32\zeta(3)\ell - \frac{88}{3}\ell^2 - \frac{8}{3}\ell^3 \right) T_F N_f \right. \\ & + \left(\frac{14419}{18} - 300\zeta(3) - 18\zeta(4) - 40\zeta(5) + \frac{893}{2}\ell - 124\zeta(3)\ell + \frac{284}{3}\ell^2 + \frac{22}{3}\ell^3 \right) C_A \\ & + \left(\frac{1613}{4} - 384\zeta(3) + 36\zeta(4) + 240\zeta(5) + \frac{691}{2}\ell \right. \\ & \left. \left. - 72\zeta(3)\ell + 105\ell^2 + 12\ell^3 \right) C_F \right] a^2 \left. \right] + O(a^3) \end{aligned} \quad (4.3)$$

$$\begin{aligned} \gamma^{S,S}(a) = d(R) & \left[2 + 10C_F a + \frac{C_F}{2} \left[(154 - 72\zeta(3)) C_A \right. \right. \\ & \left. \left. + (144\zeta(3) - 119) C_F - 32T_F N_f \right] a^2 \right] + O(a^3) . \end{aligned} \quad (4.4)$$

4.2 Vector-vector

$$\Pi_{(i)}^{V,V}(q) = q^2 \tilde{\Pi}_{(i)}^{V,V}(a) \quad (4.5)$$

$$\begin{aligned} \tilde{\Pi}_{(1)}^{V,V}(a) = -2\tilde{\Pi}_{(1)}^{V,W_2}(a) = -\tilde{\Pi}_{(1)}^{V,\partial W_2}(a) \\ = d(R) & \left[-\frac{20}{9} - \frac{4}{3}\ell + C_F \left[16\zeta(3) - \frac{55}{3} - 4\ell \right] a \right. \\ & + C_F \left[\left(\frac{7402}{81} - \frac{608}{9}\zeta(3) + \frac{88}{3}\ell - \frac{64}{3}\zeta(3)\ell + \frac{8}{3}\ell^2 \right) T_F N_f \right. \\ & + \left(\frac{1816}{9}\zeta(3) + \frac{80}{3}\zeta(5) - \frac{44215}{162} - 82\ell + \frac{176}{3}\zeta(3)\ell - \frac{22}{3}\ell^2 \right) C_A \\ & \left. \left. + \left(\frac{286}{9} + \frac{296}{3}\zeta(3) - 160\zeta(5) + 2\ell \right) C_F \right] a^2 \right] + O(a^3) \\ \tilde{\Pi}_{(2)}^{V,V}(a) = -2\tilde{\Pi}_{(2)}^{V,W_2}(a) = -\tilde{\Pi}_{(2)}^{V,\partial W_2}(a) = O(a^3) \end{aligned} \quad (4.6)$$

$$\gamma^{V,V}(a) = d(R) \left[-\frac{4}{3} - 4C_F a + \frac{C_F}{9} [18C_F - 133C_A + 44T_F N_f] a^2 \right] + O(a^3). \quad (4.7)$$

4.3 Tensor-tensor

$$\Pi_{(i)}^{T,T}(q) = q^2 \tilde{\Pi}_{(i)}^{T,T}(a) \tag{4.8}$$

$$\begin{aligned} \tilde{\Pi}_{(1)}^{T,T}(a) = & d(R) \left[-\frac{4}{9} - \frac{2}{3}\ell + C_F \left[8\zeta(3) - \frac{491}{54} - \frac{14}{9}\ell + \frac{2}{3}\ell^2 \right] a \right. \\ & + C_F \left[\left(\frac{10672}{243} - \frac{1024}{27}\zeta(3) + \frac{766}{81}\ell - \frac{32}{3}\zeta(3)\ell - \frac{8}{9}\ell^2 - \frac{8}{27}\ell^3 \right) T_F N_f \right. \\ & + \left(\frac{2732}{27}\zeta(3) - \frac{14}{3}\zeta(4) + \frac{40}{3}\zeta(5) - \frac{19427}{162} \right. \\ & \left. \left. - \frac{1771}{162}\ell + 20\zeta(3)\ell + \frac{20}{3}\ell^2 + \frac{22}{27}\ell^3 \right) C_A \right. \\ & + \left(\frac{608}{9}\zeta(3) + \frac{28}{3}\zeta(4) - 80\zeta(5) - \frac{15973}{972} - \frac{1075}{54}\ell \right. \\ & \left. \left. + \frac{8}{3}\zeta(3)\ell - \frac{43}{9}\ell^2 - \frac{4}{9}\ell^3 \right) C_F \right] a^2 \left. \right] + O(a^3) \end{aligned} \tag{4.9}$$

$$\begin{aligned} \tilde{\Pi}_{(2)}^{T,T}(a) = & d(R) \left[\frac{20}{9} + \frac{4}{3}\ell + C_F \left[\frac{593}{27} - 16\zeta(3) + \frac{28}{9}\ell - \frac{4}{3}\ell^2 \right] a \right. \\ & + C_F \left[\left(\frac{2048}{27}\zeta(3) - \frac{21328}{243} - \frac{1532}{81}\ell + \frac{64}{3}\zeta(3)\ell + \frac{16}{9}\ell^2 + \frac{16}{27}\ell^3 \right) T_F N_f \right. \\ & + \left(\frac{58075}{243} - \frac{5296}{27}\zeta(3) + \frac{28}{3}\zeta(4) - \frac{80}{3}\zeta(5) \right. \\ & \left. \left. + \frac{1771}{81}\ell - 40\zeta(3)\ell - \frac{40}{3}\ell^2 - \frac{44}{27}\ell^3 \right) C_A \right. \\ & + \left(\frac{22051}{486} - \frac{1328}{9}\zeta(3) - \frac{56}{3}\zeta(4) + 160\zeta(5) + \frac{1075}{27}\ell \right. \\ & \left. \left. - \frac{16}{3}\zeta(3)\ell + \frac{86}{9}\ell^2 + \frac{8}{9}\ell^3 \right) C_F \right] a^2 \left. \right] + O(a^3) \end{aligned} \tag{4.10}$$

$$\begin{aligned} \gamma_{(1)}^{T,T}(a) = & d(R) \left[-\frac{2}{3} - \frac{22}{9}C_F a + \frac{C_F}{162} \left[(3024\zeta(3) - 4803)C_F \right. \right. \\ & \left. \left. + (1574 - 1512\zeta(3))C_A - 16T_F N_f \right] a^2 \right] + O(a^3) \end{aligned}$$

$$\begin{aligned} \gamma_{(2)}^{T,T}(a) = & d(R) \left[\frac{4}{3} + \frac{68}{9}C_F a + \frac{C_F}{81} \left[(1512\zeta(3) + 388)C_A \right. \right. \\ & \left. \left. + (3363 - 3024\zeta(3))C_F - 200T_F N_f \right] a^2 \right] + O(a^3). \end{aligned} \tag{4.11}$$

4.4 Vector-Wilson 2

$$\Pi_{(i)}^{V,W_2}(q) = q^2 \tilde{\Pi}_{(i)}^{V,W_2}(a) \tag{4.12}$$

$$\gamma^{V,W_2}(a) = d(R) \left[\frac{2}{3} + 2C_F a + \frac{C_F}{18} \left[133C_A - 18C_F - 44T_F N_f \right] a^2 \right] + O(a^3)$$

$$\gamma^{V,\partial W_2}(a) = d(R) \left[\frac{4}{3} + 4C_F a + \frac{C_F}{9} [133C_A - 18C_F - 44T_F N_f] a^2 \right] + O(a^3) \quad (4.13)$$

4.5 Wilson 2-Wilson 2

$$\Pi_{(i)}^{W_2, W_2}(q) = (q^2)^2 \tilde{\Pi}_{(i)}^{W_2, W_2}(a) \quad (4.14)$$

$$\begin{aligned} \tilde{\Pi}_{(1)}^{W_2, W_2}(a) = & d(R) \left[\frac{12}{25} + \frac{1}{5}\ell - C_F \left[\frac{12}{5}\zeta(3) + \frac{17533}{13500} + \frac{473}{225}\ell + \frac{8}{15}\ell^2 \right] a \right. \\ & + C_F \left[\left(\frac{419327}{303750} + \frac{1816}{135}\zeta(3) + \frac{69266}{10125}\ell \right. \right. \\ & + \left. \frac{16}{5}\zeta(3)\ell + \frac{1586}{675}\ell^2 + \frac{32}{135}\ell^3 \right) T_F N_f \\ & + \left(\frac{16}{5}\zeta(4) - \frac{15838}{675}\zeta(3) - 4\zeta(5) - \frac{3541817}{1215000} - \frac{399953}{20250}\ell \right. \\ & - \left. \frac{12}{5}\zeta(3)\ell - \frac{8963}{1350}\ell^2 - \frac{88}{135}\ell^3 \right) C_A \\ & + \left(\frac{12235087}{455625} - \frac{2606}{135}\zeta(3) - \frac{32}{5}\zeta(4) + 24\zeta(5) + \frac{383653}{20250}\ell \right. \\ & \left. \left. + \frac{1448}{225}\ell^2 + \frac{128}{135}\ell^3 \right) C_F \right] a^2 \left. \right] + O(a^3) \end{aligned} \quad (4.15)$$

$$\begin{aligned} \tilde{\Pi}_{(2)}^{W_2, W_2}(a) = & d(R) \left[\frac{92}{225} + \frac{2}{15}\ell - C_F \left[\frac{8}{5}\zeta(3) + \frac{15683}{20250} + \frac{986}{675}\ell + \frac{16}{45}\ell^2 \right] a \right. \\ & + C_F \left[\left(\frac{1685066}{1366875} + \frac{3632}{405}\zeta(3) + \frac{48404}{10125}\ell \right. \right. \\ & + \left. \frac{32}{15}\zeta(3)\ell + \frac{1084}{675}\ell^2 + \frac{64}{405}\ell^3 \right) T_F N_f \\ & + \left(\frac{32}{15}\zeta(4) - \frac{29276}{2025}\zeta(3) - \frac{8}{5}\zeta(5) - \frac{7819793}{2733750} - \frac{46547}{3375}\ell \right. \\ & - \left. \frac{8}{5}\zeta(3)\ell - \frac{3061}{675}\ell^2 - \frac{176}{405}\ell^3 \right) C_A \\ & + \left(\frac{8792588}{455625} - \frac{5788}{405}\zeta(3) - \frac{64}{15}\zeta(4) + 16\zeta(5) + \frac{44597}{3375}\ell \right. \\ & \left. \left. + \frac{9008}{2025}\ell^2 + \frac{256}{405}\ell^3 \right) C_F \right] a^2 \left. \right] + O(a^3) \end{aligned} \quad (4.16)$$

$$\begin{aligned} \tilde{\Pi}_{(3)}^{W_2, W_2}(a) = & d(R) \left[\frac{17}{225} + \frac{2}{15}\ell - C_F \left[\frac{8}{5}\zeta(3) - \frac{34439}{6750} - \frac{638}{225}\ell - \frac{8}{15}\ell^2 \right] a \right. \\ & + C_F \left[\left(\frac{464}{135}\zeta(3) - \frac{6263527}{303750} - \frac{128096}{10125}\ell \right. \right. \\ & + \left. \frac{32}{15}\zeta(3)\ell - \frac{1916}{675}\ell^2 - \frac{32}{135}\ell^3 \right) T_F N_f \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{918157}{15000} - \frac{17492}{675} \zeta(3) - \frac{16}{5} \zeta(4) - \frac{8}{3} \zeta(5) + \frac{364234}{10125} \ell \right. \\
 & \left. - \frac{184}{15} \zeta(3) \ell + \frac{5389}{675} \ell^2 + \frac{88}{135} \ell^3 \right) C_A \\
 & + \left(16 \zeta(5) + \frac{32}{5} \zeta(4) - \frac{436}{135} \zeta(3) - \frac{28297949}{911250} - \frac{160709}{10125} \ell \right. \\
 & \left. - \frac{1288}{225} \ell^2 - \frac{128}{135} \ell^3 \right) C_F \Big] a^2 \Big] + O(a^3) \tag{4.17}
 \end{aligned}$$

$$\tilde{\Pi}_{(1)}^{W_2, \partial W_2}(a) = \tilde{\Pi}_{(2)}^{W_2, \partial W_2}(a) = O(a^3)$$

$$\begin{aligned}
 \tilde{\Pi}_{(3)}^{W_2, \partial W_2}(a) & = d(R) \left[\frac{10}{9} + \frac{2}{3} \ell + C_F \left[\frac{55}{6} - 8 \zeta(3) + 2 \ell \right] a \right. \\
 & + C_F \left[\left(\frac{304}{9} \zeta(3) - \frac{3701}{81} - \frac{44}{3} \ell + \frac{32}{3} \zeta(3) \ell - \frac{4}{3} \ell^2 \right) T_F N_f \right. \\
 & + \left(\frac{44215}{324} - \frac{908}{9} \zeta(3) - \frac{40}{3} \zeta(5) + 41 \ell - \frac{88}{3} \zeta(3) \ell + \frac{11}{3} \ell^2 \right) C_A \\
 & \left. \left. + \left(80 \zeta(5) - \frac{148}{3} \zeta(3) - \frac{143}{9} - \ell \right) C_F \right] a^2 \right] + O(a^3) \tag{4.18}
 \end{aligned}$$

$$\tilde{\Pi}_{(1)}^{\partial W_2, \partial W_2}(a) = \tilde{\Pi}_{(2)}^{\partial W_2, \partial W_2}(a) = O(a^3)$$

$$\tilde{\Pi}_{(3)}^{\partial W_2, \partial W_2}(a) = 2 \tilde{\Pi}_{(3)}^{W_2, \partial W_2}(a) + O(a^3) \tag{4.19}$$

$$\begin{aligned}
 \gamma_{(1)}^{W_2, W_2}(a) & = d(R) \left[\frac{1}{5} + \frac{103}{225} C_F a + \frac{C_F}{40500} \left[(259200 \zeta(3) - 65603) C_A \right. \right. \\
 & \left. \left. + (325498 - 518400 \zeta(3)) C_F + 22612 T_F N_f \right] a^2 \right] + O(a^3)
 \end{aligned}$$

$$\begin{aligned}
 \gamma_{(2)}^{W_2, W_2}(a) & = d(R) \left[\frac{2}{15} + \frac{18}{25} C_F a + \frac{C_F}{60750} \left[(26507 + 259200 \zeta(3)) C_A \right. \right. \\
 & \left. \left. + (345738 - 518400 \zeta(3)) C_F - 7828 T_F N_f \right] a^2 \right] + O(a^3)
 \end{aligned}$$

$$\begin{aligned}
 \gamma_{(3)}^{W_2, W_2}(a) & = d(R) \left[\frac{2}{15} + \frac{62}{225} C_F a + \frac{C_F}{20250} \left[(78919 - 129600 \zeta(3)) C_A \right. \right. \\
 & \left. \left. + (259200 \zeta(3) - 184754) C_F - 26276 T_F N_f \right] a^2 \right] + O(a^3)
 \end{aligned}$$

$$\gamma_{(1)}^{W_2, \partial W_2}(a) = \gamma_{(2)}^{W_2, \partial W_2}(a) = 0$$

$$\gamma_{(3)}^{W_2, \partial W_2}(a) = d(R) \left[\frac{2}{3} + 2 C_F a + \frac{C_F}{18} \left[133 C_A - 18 C_F - 44 T_F N_f \right] a^2 \right] + O(a^3)$$

$$\gamma_{(1)}^{\partial W_2, \partial W_2}(a) = \gamma_{(2)}^{\partial W_2, \partial W_2}(a) = 0$$

$$\gamma_{(3)}^{\partial W_2, \partial W_2}(a) = d(R) \left[\frac{4}{3} + 4C_F a + \frac{C_F}{9} [133C_A - 18C_F - 44T_F N_f] a^2 \right] + O(a^3). \quad (4.20)$$

4.6 Vector-Wilson 3

$$\Pi_{(i)}^{V, W_3}(q) = (q^2)^2 \tilde{\Pi}_{(i)}^{V, W_3}(a) \quad (4.21)$$

$$\begin{aligned} \tilde{\Pi}_{(1)}^{V, W_3}(a) &= -\tilde{\Pi}_{(2)}^{V, W_3}(a) \\ &= d(R) \left[\frac{31}{675} + \frac{1}{45} \ell + C_F \left[\frac{2177}{6480} - \frac{4}{15} \zeta(3) + \frac{2}{27} \ell \right] a \right. \\ &\quad + C_F \left[\left(\frac{152}{135} \zeta(3) - \frac{4070273}{2624400} - \frac{7267}{14580} \ell + \frac{16}{45} \zeta(3) \ell - \frac{11}{243} \ell^2 \right) T_F N_f \right. \\ &\quad + \left(\frac{48524449}{10497600} - \frac{1364}{405} \zeta(3) - \frac{4}{9} \zeta(5) + \frac{4051}{2916} \ell - \frac{44}{45} \zeta(3) \ell + \frac{121}{972} \ell^2 \right) C_A \\ &\quad \left. \left. + \left(\frac{8}{3} \zeta(5) - \frac{1118783}{2624400} - \frac{140}{81} \zeta(3) - \frac{187}{19440} \ell \right) C_F \right] a^2 \right] + O(a^3) \\ \tilde{\Pi}_{(1)}^{V, \partial W_3}(a) &= -\tilde{\Pi}_{(2)}^{V, \partial W_3}(a) = \frac{1}{2} \tilde{\Pi}_{(1)}^{V, \partial \partial W_3}(a) = -\frac{1}{2} \tilde{\Pi}_{(2)}^{V, \partial \partial W_3}(a) \\ &= d(R) \left[\frac{2}{27} + \frac{1}{27} \ell + C_F \left[\frac{19}{36} - \frac{4}{9} \zeta(3) + \frac{1}{9} \ell \right] a \right. \\ &\quad + C_F \left[\left(\frac{152}{81} \zeta(3) - \frac{3719}{1458} - \frac{22}{27} \ell + \frac{16}{27} \zeta(3) \ell - \frac{2}{27} \ell^2 \right) T_F N_f \right. \\ &\quad + \left(\frac{44437}{5832} - \frac{454}{81} \zeta(3) - \frac{20}{27} \zeta(5) + \frac{41}{18} \ell - \frac{44}{27} \zeta(3) \ell + \frac{11}{54} \ell^2 \right) C_A \\ &\quad \left. \left. + \left(\frac{40}{9} \zeta(5) - \frac{8}{9} - \frac{74}{27} \zeta(3) - \frac{1}{18} \ell \right) C_F \right] a^2 \right] + O(a^3) \end{aligned} \quad (4.22)$$

$$\begin{aligned} \gamma_{(1)}^{V, W_3}(a) &= -\gamma_{(2)}^{V, W_3}(a) = d(R) \left[\frac{1}{45} + \frac{13}{162} C_F a \right. \\ &\quad \left. + \frac{C_F}{116640} [27062C_A + 239C_F - 9136T_F N_f] a^2 \right] + O(a^3) \\ \gamma_{(1)}^{V, \partial W_3}(a) &= -\gamma_{(2)}^{V, \partial W_3}(a) = \frac{1}{2} \gamma_{(1)}^{V, \partial \partial W_3}(a) = -\frac{1}{2} \gamma_{(2)}^{V, \partial \partial W_3}(a) \\ &= d(R) \left[\frac{1}{27} + \frac{1}{9} C_F a + \frac{C_F}{108} [37C_A - 6C_F - 12T_F N_f] a^2 \right] + O(a^3). \end{aligned} \quad (4.23)$$

4.7 Wilson 3-Wilson 3

$$\Pi_{(i)}^{W_3, W_3}(q) = (q^2)^3 \tilde{\Pi}_{(i)}^{W_3, W_3}(a) \quad (4.24)$$

$$\begin{aligned}
 \tilde{\Pi}_{(1)}^{W_3, W_3}(a) &= d(R) \left[\frac{457}{396900} + \frac{1}{3780} \ell - C_F \left[\frac{4831049}{200037600} + \frac{1}{315} \zeta(3) + \frac{3}{196} \ell + \frac{1}{378} \ell^2 \right] a \right. \\
 &+ C_F \left[\left(\frac{254}{8505} \zeta(3) + \frac{61767749}{675126900} + \frac{45362}{694575} \ell \right. \right. \\
 &+ \left. \left. \frac{4}{945} \zeta(3) \ell + \frac{5339}{357210} \ell^2 + \frac{2}{1701} \ell^3 \right) T_F N_f \right. \\
 &+ \left(\frac{277}{23814} \zeta(3) + \frac{1}{63} \zeta(4) - \frac{1}{189} \zeta(5) - \frac{27125381251}{108020304000} \right. \\
 &- \left. \frac{17694461}{100018800} \ell + \frac{19}{945} \zeta(3) \ell - \frac{14383}{357210} \ell^2 - \frac{11}{3402} \ell^3 \right) C_A \\
 &+ \left(\frac{84198061049}{216040608000} + \frac{2}{63} \zeta(5) - \frac{2}{63} \zeta(4) - \frac{59}{945} \zeta(3) \right. \\
 &+ \left. \frac{54952451}{240045120} \ell + \frac{3959}{63504} \ell^2 + \frac{25}{3402} \ell^3 \right) C_F \left. \right] a^2 + O(a^3) \\
 \tilde{\Pi}_{(2)}^{W_3, W_3}(a) &= d(R) \left[-\frac{599}{132300} - \frac{1}{630} \ell + C_F \left[\frac{7012477}{133358400} + \frac{2}{105} \zeta(3) + \frac{5851}{158760} \ell + \frac{5}{756} \ell^2 \right] a \right. \\
 &+ C_F \left[-\left(\frac{1034}{8505} \zeta(3) + \frac{18993505}{100018800} + \frac{5117891}{33339600} \ell \right. \right. \\
 &+ \left. \left. \frac{8}{315} \zeta(3) \ell + \frac{2189}{59535} \ell^2 + \frac{5}{1701} \ell^3 \right) T_F N_f \right. \\
 &+ \left(\frac{6793}{59535} \zeta(3) - \frac{5}{126} \zeta(4) + \frac{2}{63} \zeta(5) + \frac{269779943}{533433600} \right. \\
 &+ \left. \frac{3463261}{8334900} \ell - \frac{1}{105} \zeta(3) \ell + \frac{94321}{952560} \ell^2 + \frac{55}{6804} \ell^3 \right) C_A \\
 &+ \left(\frac{6619}{34020} \zeta(3) + \frac{5}{63} \zeta(4) - \frac{4}{21} \zeta(5) - \frac{5595369371}{5761082880} \right. \\
 &- \left. \frac{95885941}{160030080} \ell - \frac{20453}{127008} \ell^2 - \frac{125}{6804} \ell^3 \right) C_F \left. \right] a^2 + O(a^3) \\
 \tilde{\Pi}_{(3)}^{W_3, W_3}(a) &= d(R) \left[\frac{4051}{396900} + \frac{13}{3780} \ell - C_F \left[\frac{5103787}{200037600} + \frac{13}{315} \zeta(3) + \frac{63503}{2381400} \ell + \frac{41}{5670} \ell^2 \right] a \right. \\
 &+ C_F \left[\left(\frac{5594}{25515} \zeta(3) - \frac{9014474417}{270050760000} + \frac{3187379}{41674500} \ell \right. \right. \\
 &+ \left. \left. \frac{52}{945} \zeta(3) \ell + \frac{109283}{3572100} \ell^2 + \frac{82}{25515} \ell^3 \right) T_F N_f \right. \\
 &+ \left(\frac{16759291981}{154314720000} - \frac{364031}{893025} \zeta(3) + \frac{41}{945} \zeta(4) - \frac{13}{189} \zeta(5) \right. \\
 &- \left. \frac{230532091}{1000188000} \ell - \frac{61}{945} \zeta(3) \ell - \frac{1261123}{14288400} \ell^2 - \frac{451}{51030} \ell^3 \right) C_A \\
 &+ \left(\frac{221977783933}{1080203040000} + \frac{26}{63} \zeta(5) - \frac{82}{945} \zeta(4) - \frac{8251}{25515} \zeta(3) \right. \\
 &+ \left. \frac{19415531}{111132000} \ell + \frac{159653}{2381400} \ell^2 + \frac{521}{51030} \ell^3 \right) C_F \left. \right] a^2 + O(a^3)
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\Pi}_{(4)}^{W_3, W_3}(a) = & d(R) \left[-\frac{233}{26460} - \frac{1}{252} \ell + C_F \left[\frac{1}{21} \zeta(3) - \frac{6186559}{666792000} + \frac{6283}{264600} \ell + \frac{31}{3780} \ell^2 \right] a \right. \\
 & + C_F \left[\left(\frac{577820077}{5000940000} - \frac{2144}{8505} \zeta(3) - \frac{2806243}{55566000} \ell \right. \right. \\
 & \left. \left. - \frac{4}{63} \zeta(3) \ell - \frac{8921}{297675} \ell^2 - \frac{31}{8505} \ell^3 \right) T_F N_f \right. \\
 & + \left(\frac{314851}{595350} \zeta(3) - \frac{31}{630} \zeta(4) + \frac{5}{63} \zeta(5) - \frac{5651256311}{17146080000} \right. \\
 & \left. + \frac{28483291}{166698000} \ell + \frac{8}{105} \zeta(3) \ell + \frac{426019}{4762800} \ell^2 + \frac{341}{34020} \ell^3 \right) C_A \\
 & + \left(\frac{11023}{34020} \zeta(3) + \frac{31}{315} \zeta(4) - \frac{10}{21} \zeta(5) + \frac{95645127727}{720135360000} \right. \\
 & \left. + \frac{64654361}{4000752000} \ell - \frac{256373}{9525600} \ell^2 - \frac{271}{34020} \ell^3 \right) C_F \left. \right] a^2 \Big] + O(a^3) \tag{4.25}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\Pi}_{(1)}^{W_3, \partial W_3}(a) = & d(R) \left[-\frac{1}{675} - \frac{1}{1620} \ell + C_F \left[\frac{1}{135} \zeta(3) - \frac{24571}{3499200} - \frac{1}{810} \ell \right] a \right. \\
 & + C_F \left[\left(\frac{3011809}{94478400} - \frac{38}{1215} \zeta(3) + \frac{23999}{2624400} \ell - \frac{4}{405} \zeta(3) \ell + \frac{31}{43740} \ell^2 \right) T_F N_f \right. \\
 & + \left(\frac{329}{3645} \zeta(3) + \frac{1}{81} \zeta(5) - \frac{36851477}{377913600} \right. \\
 & \left. - \frac{66631}{2624400} \ell + \frac{11}{405} \zeta(3) \ell - \frac{341}{174960} \ell^2 \right) C_A \\
 & + \left(\frac{151}{3645} \zeta(3) - \frac{2}{27} \zeta(5) + \frac{175559}{188956800} \right. \\
 & \left. - \frac{37657}{3499200} \ell - \frac{8}{3645} \ell^2 \right) C_F \left. \right] a^2 \Big] + O(a^3)
 \end{aligned}$$

$$\tilde{\Pi}_{(2)}^{W_3, \partial W_3}(a) = O(a^3)$$

$$\begin{aligned}
 \tilde{\Pi}_{(3)}^{W_3, \partial W_3}(a) = & d(R) \left[\frac{26}{2025} + \frac{7}{1620} \ell - C_F \left[\frac{255937}{17496000} + \frac{7}{135} \zeta(3) + \frac{239}{6075} \ell + \frac{4}{405} \ell^2 \right] a \right. \\
 & + C_F \left[\left(\frac{5571143}{2361960000} + \frac{1022}{3645} \zeta(3) + \frac{1622549}{13122000} \ell \right. \right. \\
 & \left. \left. + \frac{28}{405} \zeta(3) \ell + \frac{9601}{218700} \ell^2 + \frac{16}{3645} \ell^3 \right) T_F N_f \right. \\
 & + \left(\frac{170483597}{9447840000} - \frac{332}{675} \zeta(3) + \frac{8}{135} \zeta(4) - \frac{7}{81} \zeta(5) \right. \\
 & \left. - \frac{4693921}{13122000} \ell - \frac{29}{405} \zeta(3) \ell - \frac{108491}{874800} \ell^2 - \frac{44}{3645} \ell^3 \right) C_A \\
 & + \left(\frac{2525581969}{4723920000} + \frac{14}{27} \zeta(5) - \frac{16}{135} \zeta(4) - \frac{1598}{3645} \zeta(3) \right. \\
 & \left. + \frac{6610253}{17496000} \ell + \frac{764}{6075} \ell^2 + \frac{64}{3645} \ell^3 \right) C_F \left. \right] a^2 \Big] + O(a^3)
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\Pi}_{(4)}^{W_3, \partial W_3}(a) &= \tilde{\Pi}_{(4)}^{\partial W_3, \partial W_3}(a) \\
 &= d(R) \left[-\frac{1}{75} - \frac{1}{180}\ell + C_F \left[\frac{17533}{486000} + \frac{1}{15}\zeta(3) + \frac{473}{8100}\ell + \frac{2}{135}\ell^2 \right] a \right. \\
 &\quad + C_F \left[-\left(\frac{419327}{10935000} + \frac{454}{1215}\zeta(3) + \frac{34633}{182250}\ell \right. \right. \\
 &\quad \left. \left. + \frac{4}{45}\zeta(3)\ell + \frac{793}{12150}\ell^2 + \frac{8}{1215}\ell^3 \right) T_F N_f \right. \\
 &\quad + \left(\frac{3541817}{43740000} + \frac{7919}{12150}\zeta(3) - \frac{4}{45}\zeta(4) + \frac{1}{9}\zeta(5) \right. \\
 &\quad \left. + \frac{399953}{729000}\ell + \frac{1}{15}\zeta(3)\ell + \frac{8963}{48600}\ell^2 + \frac{22}{1215}\ell^3 \right) C_A \\
 &\quad + \left(\frac{1303}{2430}\zeta(3) - \frac{2}{3}\zeta(5) + \frac{8}{45}\zeta(4) - \frac{12235087}{16402500} - \frac{383653}{729000}\ell \right. \\
 &\quad \left. - \frac{362}{2025}\ell^2 - \frac{32}{1215}\ell^3 \right) C_F \left. \right] a^2 \Big] + O(a^3) \tag{4.26}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\Pi}_{(1)}^{W_3, \partial \partial W_3}(a) &= -\tilde{\Pi}_{(3)}^{W_3, \partial \partial W_3}(a) \\
 &= d(R) \left[-\frac{2}{675} - \frac{1}{810}\ell + C_F \left[\frac{2}{135}\zeta(3) - \frac{6971}{349920} - \frac{1}{243}\ell \right] a \right. \\
 &\quad + C_F \left[\left(\frac{4088713}{47239200} - \frac{76}{1215}\zeta(3) + \frac{7267}{2624400}\ell - \frac{8}{405}\zeta(3)\ell + \frac{11}{4374}\ell^2 \right) T_F N_f \right. \\
 &\quad + \left(\frac{682}{3645}\zeta(3) + \frac{2}{81}\zeta(5) - \frac{48746669}{188956800} \right. \\
 &\quad \left. - \frac{4051}{52488}\ell + \frac{22}{405}\zeta(3)\ell - \frac{121}{17496}\ell^2 \right) C_A \\
 &\quad \left. + \left(\frac{70}{729}\zeta(3) - \frac{4}{27}\zeta(5) + \frac{2236871}{94478400} + \frac{187}{349920}\ell \right) C_F \right] a^2 \Big] + O(a^3) \\
 \tilde{\Pi}_{(2)}^{W_3, \partial \partial W_3}(a) &= \tilde{\Pi}_{(4)}^{W_3, \partial \partial W_3}(a) = O(a^3) \tag{4.27}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\Pi}_{(1)}^{\partial W_3, \partial W_3}(a) &= d(R) \left[-\frac{7}{2916} - \frac{1}{972}\ell + C_F \left[\frac{1}{81}\zeta(3) - \frac{3643}{291600} - \frac{11}{4860}\ell \right] a \right. \\
 &\quad + C_F \left[\left(\frac{9796}{164025} - \frac{38}{729}\zeta(3) + \frac{1961}{109350}\ell - \frac{4}{243}\zeta(3)\ell + \frac{11}{7290}\ell^2 \right) T_F N_f \right. \\
 &\quad + \left(\frac{1111}{7290}\zeta(3) + \frac{5}{243}\zeta(5) - \frac{237541}{1312200} \right. \\
 &\quad \left. - \frac{21901}{437400}\ell + \frac{11}{243}\zeta(3)\ell - \frac{121}{29160}\ell^2 \right) C_A \\
 &\quad + \left(\frac{169}{2430}\zeta(3) - \frac{10}{81}\zeta(5) + \frac{54931}{3936600} \right. \\
 &\quad \left. - \frac{461}{48600}\ell - \frac{8}{3645}\ell^2 \right) C_F \left. \right] a^2 \Big] + O(a^3)
 \end{aligned}$$

$$\tilde{\Pi}_{(2)}^{\partial W_3, \partial W_3}(a) = O(a^3)$$

$$\begin{aligned} \tilde{\Pi}_{(3)}^{\partial W_3, \partial W_3}(a) = & d(R) \left[\frac{1003}{72900} + \frac{23}{4860} \ell - C_F \left[\frac{13351}{1458000} + \frac{23}{405} \zeta(3) + \frac{931}{24300} \ell + \frac{4}{405} \ell^2 \right] a \right. \\ & + C_F \left[\left(\frac{122}{405} \zeta(3) - \frac{104507}{4100625} + \frac{62801}{546750} \ell \right. \right. \\ & + \left. \frac{92}{1215} \zeta(3) \ell + \frac{1571}{36450} \ell^2 + \frac{16}{3645} \ell^3 \right) T_F N_f \\ & + \left(\frac{1110523}{10935000} - \frac{6731}{12150} \zeta(3) + \frac{8}{135} \zeta(4) - \frac{23}{243} \zeta(5) \right. \\ & - \left. \frac{728341}{2187000} \ell - \frac{109}{1215} \zeta(3) \ell - \frac{17761}{145800} \ell^2 - \frac{44}{3645} \ell^3 \right) C_A \\ & + \left(\frac{51334453}{98415000} + \frac{46}{81} \zeta(5) - \frac{16}{135} \zeta(4) - \frac{3401}{7290} \zeta(3) + \frac{91499}{243000} \ell \right. \\ & \left. \left. + \frac{764}{6075} \ell^2 + \frac{64}{3645} \ell^3 \right) C_F \right] a^2 \Big] + O(a^3) \end{aligned} \quad (4.28)$$

$$\begin{aligned} \tilde{\Pi}_{(1)}^{\partial W_3, \partial \partial W_3}(a) = & - \tilde{\Pi}_{(3)}^{\partial W_3, \partial \partial W_3}(a) = \frac{1}{2} \tilde{\Pi}_{(1)}^{\partial \partial W_3, \partial \partial W_3}(a) = - \frac{1}{2} \tilde{\Pi}_{(3)}^{\partial \partial W_3, \partial \partial W_3}(a) \\ = & d(R) \left[- \frac{7}{1458} - \frac{1}{486} \ell - C_F \left[\frac{59}{1944} - \frac{2}{81} \zeta(3) + \frac{1}{162} \ell \right] a \right. \\ & + C_F \left[\left(\frac{3733}{26244} - \frac{76}{729} \zeta(3) + \frac{11}{243} \ell - \frac{8}{243} \zeta(3) \ell + \frac{1}{243} \ell^2 \right) T_F N_f \right. \\ & + \left(\frac{10}{243} \zeta(5) + \frac{227}{729} \zeta(3) - \frac{44615}{104976} - \frac{41}{324} \ell + \frac{22}{243} \zeta(3) \ell - \frac{11}{972} \ell^2 \right) C_A \\ & \left. \left. + \left(\frac{145}{2916} + \frac{37}{243} \zeta(3) - \frac{20}{81} \zeta(5) + \frac{1}{324} \ell \right) C_F \right] a^2 \right] + O(a^3) \\ \tilde{\Pi}_{(2)}^{\partial W_3, \partial \partial W_3}(a) = & \tilde{\Pi}_{(4)}^{\partial W_3, \partial \partial W_3}(a) = \tilde{\Pi}_{(2)}^{\partial \partial W_3, \partial \partial W_3}(a) = \tilde{\Pi}_{(4)}^{\partial \partial W_3, \partial \partial W_3}(a) = O(a^3) \end{aligned} \quad (4.29)$$

$$\begin{aligned} \gamma_{(1)}^{W_3, W_3}(a) = & d(R) \left[\frac{1}{3780} + \frac{101}{59535} C_F a + \frac{C_F}{600112800} \left[(19051200 \zeta(3) - 5588635) C_A \right. \right. \\ & \left. \left. + (30839406 - 38102400 \zeta(3)) C_F + 1443956 T_F N_f \right] a^2 \right] + O(a^3) \end{aligned}$$

$$\begin{aligned} \gamma_{(2)}^{W_3, W_3}(a) = & d(R) \left[- \frac{1}{630} - \frac{139}{158760} C_F a + \frac{C_F}{800150400} \left[(34630242 - 63504000 \zeta(3)) C_A \right. \right. \\ & \left. \left. + (127008000 - 94674835 \zeta(3)) C_F - 11046528 T_F N_f \right] a^2 \right] + O(a^3) \end{aligned}$$

$$\begin{aligned} \gamma_{(3)}^{W_3, W_3}(a) = & d(R) \left[\frac{13}{3780} + \frac{13421}{793800} C_F a + \frac{C_F}{2000376000} \left[(32330861 + 173577600 \zeta(3)) C_A \right. \right. \\ & \left. \left. + (200930804 - 347155200 \zeta(3)) C_F - 7817764 T_F N_f \right] a^2 \right] + O(a^3) \end{aligned}$$

$$\begin{aligned}
 \gamma_{(4)}^{W_3, W_3}(a) &= d(R) \left[-\frac{1}{252} - \frac{7649}{793800} C_F a \right. \\
 &\quad \left. + \frac{C_F}{4000752000} \left[(51833522 - 393724800\zeta(3)) C_A \right. \right. \\
 &\quad \left. \left. + (787449600\zeta(3) - 405771337) C_F - 24183088 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(1)}^{W_3, \partial W_3}(a) &= d(R) \left[-\frac{1}{1620} - \frac{41}{29160} C_F a + \frac{C_F}{20995200} \left[68041 C_F \right. \right. \\
 &\quad \left. \left. - 40406 C_A + 13264 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(2)}^{W_3, \partial W_3}(a) &= O(a^3) \\
 \gamma_{(3)}^{W_3, \partial W_3}(a) &= d(R) \left[\frac{7}{1620} + \frac{3121}{145800} C_F a + \frac{C_F}{104976000} \left[(1527166 + 12441600\zeta(3)) C_A \right. \right. \\
 &\quad \left. \left. + (16178419 - 24883200\zeta(3)) C_F - 461264 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(4)}^{W_3, \partial W_3}(a) &= d(R) \left[-\frac{1}{180} - \frac{103}{8100} C_F a + \frac{C_F}{1458000} \left[(65603 - 259200\zeta(3)) C_A \right. \right. \\
 &\quad \left. \left. + (518400\zeta(3) - 325498) C_F - 22612 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(1)}^{W_3, \partial \partial W_3}(a) &= -\gamma_{(3)}^{W_3, \partial \partial W_3}(a) \\
 &= -d(R) \left[\frac{1}{810} + \frac{13}{2916} C_F a + \frac{C_F}{2099520} \left[22222 C_A + 139 C_F - 7376 T_F N_f \right] a^2 \right] \\
 &\quad + O(a^3) \\
 \gamma_{(2)}^{W_3, \partial \partial W_3}(a) &= \gamma_{(4)}^{W_3, \partial \partial W_3}(a) = O(a^3) \\
 \gamma_{(1)}^{\partial W_3, \partial W_3}(a) &= -d(R) \left[\frac{1}{972} + \frac{11}{4860} C_F a + \frac{C_F}{874800} \left[3729 C_A - 4214 C_F - 1116 T_F N_f \right] a^2 \right] \\
 &\quad + O(a^3) \\
 \gamma_{(2)}^{\partial W_3, \partial W_3}(a) &= O(a^3) \\
 \gamma_{(3)}^{\partial W_3, \partial W_3}(a) &= d(R) \left[\frac{23}{4860} + \frac{541}{24300} C_F a + \frac{C_F}{4374000} \left[(73859 + 518400\zeta(3)) C_A \right. \right. \\
 &\quad \left. \left. + (667206 - 1036800\zeta(3)) C_F - 22036 T_F N_f \right] a^2 \right] + O(a^3)
 \end{aligned}$$

$$\begin{aligned}
 \gamma_{(4)}^{\partial W_3, \partial W_3}(a) &= d(R) \left[-\frac{1}{180} - \frac{103}{8100} C_F a + \frac{C_F}{1458000} \left[(65603 - 259200\zeta(3)) C_A \right. \right. \\
 &\quad \left. \left. + (518400\zeta(3) - 325498) C_F - 22612 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(1)}^{\partial W_3, \partial \partial W_3}(a) &= -\gamma_{(3)}^{\partial W_3, \partial \partial W_3}(a) = \frac{1}{2} \gamma_{(1)}^{\partial \partial W_3, \partial \partial W_3}(a) = -\frac{1}{2} \gamma_{(3)}^{\partial \partial W_3, \partial \partial W_3}(a) \\
 &= -d(R) \left[\frac{1}{486} + \frac{1}{162} C_F a + \frac{C_F}{5832} \left[89 C_A - 18 C_F - 28 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(2)}^{\partial W_3, \partial \partial W_3}(a) &= \gamma_{(4)}^{\partial W_3, \partial \partial W_3}(a) = \gamma_{(2)}^{\partial \partial W_3, \partial \partial W_3}(a) = \gamma_{(4)}^{\partial \partial W_3, \partial \partial W_3}(a) = O(a^3). \tag{4.30}
 \end{aligned}$$

4.8 Tensor-transversity 2

$$\Pi_{(i)}^{T, T_2}(q) = (q^2)^2 \tilde{\Pi}_{(i)}^{T, T_2}(a) \tag{4.31}$$

$$\begin{aligned}
 \tilde{\Pi}_{(1)}^{T, T_2}(a) &= d(R) \left[\frac{2}{9} + \frac{1}{3} \ell + C_F \left[\frac{491}{108} - 4\zeta(3) + \frac{7}{9} \ell - \frac{1}{3} \ell^2 \right] a \right. \\
 &\quad + C_F \left[\left(\frac{512}{27} \zeta(3) - \frac{5336}{243} - \frac{383}{41} \ell + \frac{16}{3} \zeta(3) \ell + \frac{4}{9} \ell^2 + \frac{4}{27} \ell^3 \right) T_F N_f \right. \\
 &\quad + \left(\frac{19427}{324} - \frac{1366}{27} \zeta(3) + \frac{7}{3} \zeta(4) - \frac{20}{3} \zeta(5) + \frac{1771}{324} \ell \right. \\
 &\quad \left. \left. - 10\zeta(3) \ell - \frac{10}{3} \ell^2 - \frac{11}{27} \ell^3 \right) C_A \right. \\
 &\quad + \left(\frac{15973}{1944} - \frac{304}{9} \zeta(3) - \frac{14}{3} \zeta(4) + 40\zeta(5) + \frac{1075}{108} \ell - \frac{4}{3} \zeta(3) \ell \right. \\
 &\quad \left. \left. + \frac{1075}{108} \ell + \frac{43}{18} \ell^2 + \frac{2}{9} \ell^3 \right) C_F \right] a^2 \left. \right] + O(a^3) \tag{4.32}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\Pi}_{(2)}^{T, T_2}(a) &= -2\tilde{\Pi}_{(3)}^{T, T_2}(a) + O(a^3) \\
 &= d(R) \left[\frac{20}{27} + \frac{2}{9} \ell + C_F \left[\frac{803}{162} - \frac{8}{3} \zeta(3) + \frac{14}{27} \ell - \frac{2}{9} \ell^2 \right] a \right. \\
 &\quad + C_F \left[\left(\frac{1024}{81} \zeta(3) - \frac{32368}{2187} - \frac{766}{243} \ell + \frac{32}{9} \zeta(3) \ell + \frac{8}{27} \ell^2 + \frac{8}{81} \ell^3 \right) T_F N_f \right. \\
 &\quad + \left(\frac{180043}{4374} - \frac{836}{27} \zeta(3) + \frac{14}{9} \zeta(4) - \frac{40}{9} \zeta(5) + \frac{1771}{486} \ell \right. \\
 &\quad \left. \left. - \frac{20}{3} \zeta(3) \ell - \frac{20}{9} \ell^2 - \frac{22}{81} \ell^3 \right) C_A \right. \\
 &\quad + \left(\frac{9911}{972} - \frac{2272}{81} \zeta(3) - \frac{28}{9} \zeta(4) + \frac{80}{3} \zeta(5) + \frac{1075}{162} \ell \right. \\
 &\quad \left. \left. - \frac{8}{9} \zeta(3) \ell + \frac{43}{27} \ell^2 + \frac{4}{27} \ell^3 \right) C_F \right] a^2 \left. \right] + O(a^3) \tag{4.33}
 \end{aligned}$$

$$\tilde{\Pi}_{(4)}^{T, T_2}(a) = + O(a^3) \tag{4.34}$$

$$\tilde{\Pi}_{(i)}^{T, \partial T_2}(a) = 2\tilde{\Pi}_{(i)}^{T, T_2}(a) + O(a^3) \tag{4.35}$$

$$\begin{aligned}
 \gamma_{(1)}^{T,T_2}(a) &= d(R) \left[\frac{1}{3} + \frac{11}{9} C_F a + \frac{C_F}{324} \left[(1512\zeta(3) - 1574) C_A \right. \right. \\
 &\quad \left. \left. + (4803 - 3024\zeta(3)) C_F + 16 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(2)}^{T,T_2}(a) &= d(R) \left[\frac{2}{9} + 2 C_F a + \frac{C_F}{486} \left[(3218 + 1512\zeta(3)) C_A \right. \right. \\
 &\quad \left. \left. + (1203 - 3024\zeta(3)) C_F - 400 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(3)}^{T,T_2}(a) &= -d(R) \left[\frac{1}{9} + C_F a + \frac{C_F}{972} \left[(3218 + 1512\zeta(3)) C_A \right. \right. \\
 &\quad \left. \left. + (1203 - 3024\zeta(3)) C_F - 400 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(4)}^{T,T_2}(a) &= O(a^3) \\
 \gamma_{(1)}^{T,\partial T_2}(a) &= d(R) \left[\frac{2}{3} + \frac{22}{9} C_F a + \frac{C_F}{162} \left[(1512\zeta(3) - 1574) C_A \right. \right. \\
 &\quad \left. \left. + (4803 - 3024\zeta(3)) C_F + 16 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(2)}^{T,\partial T_2}(a) &= d(R) \left[\frac{4}{9} + 4 C_F a + \frac{C_F}{243} \left[(3218 + 1512\zeta(3)) C_A \right. \right. \\
 &\quad \left. \left. + (1203 - 3024\zeta(3)) C_F - 400 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(3)}^{T,\partial T_2}(a) &= -d(R) \left[\frac{2}{9} + 2 C_F a + \frac{C_F}{486} \left[(3218 + 1512\zeta(3)) C_A \right. \right. \\
 &\quad \left. \left. + (1203 - 3024\zeta(3)) C_F - 400 T_F N_f \right] a^2 \right] + O(a^3) \\
 \gamma_{(4)}^{T,\partial T_2}(a) &= O(a^3) .
 \end{aligned} \tag{4.36}$$

Finally, we note that in addition to the various checks we have mentioned so far, our FORM code was written in such a way that only the Feynman rules for the operators and projectors needed to be input. The MINCER integration code and its interface with the QGRAF set of Feynman diagrams forms the *same* central block module of the programme. In this way our approach was designed in order to minimize the potential places where errors could creep into the overall computer algebra computation. In this respect we have not in fact derived new Feynman rules for the parent operators V , T , W_2 , W_3 or T_2 but imported those used in the programmes which underlay the results of [21, 22]. For the remaining total derivative operators it is evident from the consistency, say, in the relations of their anomalous dimensions with those without the derivatives that their FORM Feynman

rule module is not inconsistent.

5 Tensor current R -ratio

Our final exercise is to derive the R -ratio for the tensor current to complete the evaluation for all the quark bilinear currents. As for S and V it can simply be derived from the current correlator by using

$$R_{(i)}^T(a) = \frac{1}{2\pi s} \text{Im} \left(\Pi_{(i)}^T(-s - i\varepsilon) \right) \quad (5.1)$$

where ε indicates the usual shift away from the real axis to avoid ambiguity. Unlike S there are two Lorentz tensor components and for the moment we assume there are two R -ratios. The case V has in principle two similar channels but due to gauge symmetry there is no contribution in the longitudinal piece of the decomposition. With this definition and our results (4.9) and (4.10) we find

$$\begin{aligned} R_{(1)}^T(s) &= -d(R) \left[\frac{1}{3} + \left(\frac{7}{9} - \frac{2}{3}\bar{\ell} \right) C_F a + \left(\left(\frac{16}{3}\zeta(3) - \frac{383}{81} - \frac{4}{27}\pi^2 + \frac{8}{9}\bar{\ell} + \frac{4}{9}\bar{\ell}^2 \right) C_F \right. \right. \\ &\quad + \left. \left(\frac{1771}{324} - 10\zeta(3) + \frac{11}{27}\pi^2 - \frac{20}{3}\bar{\ell} - \frac{11}{9}\bar{\ell}^2 \right) C_F C_A \right. \\ &\quad \left. + \left(\frac{1075}{108} - \frac{4}{3}\zeta(3) - \frac{2}{9}\pi^2 + \frac{43}{9}\bar{\ell} + \frac{2}{3}\bar{\ell}^2 \right) C_F^2 \right] a^2 + O(a^3) \\ R_{(2)}^T(s) &= d(R) \left[\frac{2}{3} + \left(\frac{14}{9} - \frac{4}{3}\bar{\ell} \right) C_F a + \left(\left(\frac{32}{3}\zeta(3) - \frac{766}{81} - \frac{8}{27}\pi^2 + \frac{16}{9}\bar{\ell} + \frac{8}{9}\bar{\ell}^2 \right) C_F \right. \right. \\ &\quad + \left. \left(\frac{1771}{162} - 20\zeta(3) + \frac{22}{27}\pi^2 - \frac{40}{3}\bar{\ell} - \frac{22}{9}\bar{\ell}^2 \right) C_F C_A \right. \\ &\quad \left. + \left(\frac{1075}{54} - \frac{8}{3}\zeta(3) - \frac{4}{9}\pi^2 + \frac{86}{9}\bar{\ell} + \frac{4}{3}\bar{\ell}^2 \right) C_F^2 \right] a^2 + O(a^3) \end{aligned} \quad (5.2)$$

where

$$\bar{\ell} = \ln \left(\frac{\mu^2}{s} \right). \quad (5.3)$$

From these it is evident to see that there is a simple relationship to three loops between both channels which is

$$R_{(2)}^T(s) = -2R_{(1)}^T(s) + O(a^3). \quad (5.4)$$

Whilst the full expressions for (4.9) and (4.10) are different and do not satisfy an analogous relation, the behaviour of the ℓ terms do which is the origin for the result (5.4). Consequently, if we now include the Lorentz tensors of the projection basis we can write down the Lorentz tensor dependence of the R -ratio as one would have derived it directly from $\Pi_{\mu_1\mu_2\nu_1\nu_2}^{T,T}(q^2)$ if we had not had the problem of focusing on scalar amplitudes in order to perform the MINCER calculations. Therefore, we have

$$R_{\mu_1\mu_2\nu_1\nu_2}^T(s) = \left[\tilde{P}_{\mu_1\nu_1}(q)\tilde{P}_{\mu_2\nu_2}(q) - \tilde{P}_{\mu_1\nu_2}(q)\tilde{P}_{\mu_2\nu_1}(q) \right] R_{(1)}^T(s) \quad (5.5)$$

where we have introduced the common tensor structure

$$\tilde{P}^{\mu\nu}(p) = \eta^{\mu\nu} - \frac{2p^\mu p^\nu}{p^2}. \quad (5.6)$$

The appearance of this tensor structure is akin to that for case V where the longitudinal piece is absent. Put another way this form would have emerged directly if we had chosen our Lorentz tensor basis in a more erudite fashion. For completeness, we have numerically evaluated the amplitude for the colour group $SU(3)$ similar to appendix B. We have

$$\begin{aligned}
 R_{(1)}^T(s) = & -d(R) [0.333333 + [1.037037 - 0.888888\bar{\ell}] a \\
 & + [0.812771 + 0.146941N_f + (-18.172840 + 0.592593N_f)\bar{\ell} \\
 & + (-3.703704 + 0.296296N_f)\bar{\ell}^2] a^2] + O(a^3). \tag{5.7}
 \end{aligned}$$

For example, to see the convergence behaviour in relation to the expressions for S and V for three quark flavours when $s = \mu^2$ we have

$$R_{(1)}^T(\mu^2) \Big|_{N_f=3} = -3 [0.333333 + 1.037037a + 1.253594a^2] + O(a^3). \tag{5.8}$$

6 Discussion

We conclude with brief remarks since the main goal of the exercise to determine the finite parts of various operator correlation function to $O(a^2)$ in the \overline{MS} scheme has clearly been achieved. It extends the work of [26–28]. One novel feature was the need to properly account for the operator mixing into total derivative operators for the flavour non-singlet twist-2 operators used in deep inelastic scattering. The mixing matrix has been deduced for several low moments but to extend these to moments $n \geq 4$ for arbitrary n to even two loops would seem to be excluded at this stage. For instance, the calculational machinery on a par with MINCER is unfortunately not available. Whilst the main obstacle is the inability to disentangle the relations between counterterms one way through could be to embed the operators in higher leg Green’s functions. Whilst this, in principle, will give more relations between the counterterms there is again the problem of lack of calculational machinery. Indeed with more legs with independent momenta any nullification of external momenta has the additional potential problem of introducing spurious infrared singularities. These would have to be properly treated using, say, infrared rearrangement to be confident in the correctness of the final counterterm relations. However, since the main problem here was motivated by the need to provide only *low* moment flavour non-singlet information for lattice computations, this is a problem which is left for future consideration.

Acknowledgments

The author thanks Dr P.E.L. Rakow, Dr R. Horsley and Prof. A. Vogt for valuable discussions and especially the former for a careful reading of the manuscript.

A Projectors

In this appendix we record the explicit forms of the tensors into which the various correlation functions are decomposed. For each sector we also record the matrix \mathcal{M}^{ij} used to

project out each individual component of the decomposition. The matrix \mathcal{M}_{kl}^{ij} is derived by first constructing the matrix \mathcal{N}_{kl}^{ij} where k and l label the projectors, which is defined by

$$\mathcal{N}_{kl}^{ij} = \mathcal{P}_{(k)\{\mu_1 \dots \mu_{n_i} | \nu_1 \dots \nu_{n_j}\}}^{ij}(q) \mathcal{P}_{(l)}^{ij \{\mu_1 \dots \mu_{n_i} | \nu_1 \dots \nu_{n_j}\}}(q) \quad (\text{A.1})$$

where there is no sum over the i and j . The elements of this matrix are polynomials in the dimension d due to the contraction of the Lorentz indices. Finally, \mathcal{M}^{ij} is the inverse of \mathcal{N}^{ij} . Once \mathcal{M}^{ij} is specified then to project out, say, the k th piece of the tensor correlation function, one multiplies it by the projector

$$\sum_{l=1}^{n_{ij}} \mathcal{M}_{kl}^{ij} \mathcal{P}_{(l)\{\mu_1 \dots \mu_{n_i} | \nu_1 \dots \nu_{n_j}\}}^{ij}(q) \quad (\text{A.2})$$

where there is no sum over the labels $\{ij\}$. The method we have used to construct the tensor basis, which of course is not unique, is to first write down the complete set of tensors built from the metric, $\eta_{\mu\nu}$, and the momentum, q_μ , which have the same number of free indices as the operator correlation function of interest. Each of these independent tensors is then multiplied by a different label and then the Lorentz symmetry properties of the two operators in the correlation function are enforced on the sum of all independent tensors. This provides a set of linear equations for the labels which is fewer in number than the total number of original labels. Solving these equations reduces the number of independent labels, and hence independent combinations of the individual tensors, producing the tensor basis as enumerated in table 1. Therefore, it remains to list the relevant explicit expressions for the various sectors as:

A.1 Vector-vector

$$\mathcal{P}_{(1)\{\mu|\nu\}}^{V,V}(q) = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}, \quad \mathcal{P}_{(2)\{\mu|\nu\}}^{V,V}(q) = \frac{q_\mu q_\nu}{q^2} \quad (\text{A.3})$$

$$\mathcal{M}^{V,V} = \frac{1}{(d-1)q^2} \begin{pmatrix} 1 & 0 \\ 0 & d-1 \end{pmatrix}. \quad (\text{A.4})$$

A.2 Tensor-tensor

$$\begin{aligned} \mathcal{P}_{(1)\{\mu\nu|\sigma\rho\}}^{T,T}(q) &= \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma} \\ \mathcal{P}_{(2)\{\mu\nu|\sigma\rho\}}^{T,T}(q) &= \eta_{\mu\sigma} \frac{q_\nu q_\rho}{q^2} - \eta_{\mu\rho} \frac{q_\nu q_\sigma}{q^2} - \eta_{\nu\sigma} \frac{q_\mu q_\rho}{q^2} + \eta_{\nu\rho} \frac{q_\mu q_\sigma}{q^2} \end{aligned} \quad (\text{A.5})$$

$$\mathcal{M}^{T,T} = \frac{1}{4(d-1)(d-2)q^2} \begin{pmatrix} 2 & -2 \\ -2 & d \end{pmatrix}. \quad (\text{A.6})$$

A.3 Vector-Wilson 2

$$\mathcal{P}_{(1)\{\mu|\sigma\rho\}}^{V,W_2}(q) = \left[\eta_{\mu\sigma} q_\rho + \eta_{\mu\rho} q_\sigma - 2 \frac{q_\mu q_\rho q_\sigma}{q^2} \right] \frac{1}{q^2}$$

$$\mathcal{P}_{(2)\{\mu|\sigma\rho\}}^{V,W_2}(q) = \left[\eta_{\sigma\rho}q_\mu - d \frac{q_\mu q_\rho q_\sigma}{q^2} \right] \frac{1}{q^2} \quad (\text{A.7})$$

$$\mathcal{M}^{V,W_2} = \frac{1}{2d(d-1)q^2} \begin{pmatrix} d & 0 \\ 0 & 2 \end{pmatrix}. \quad (\text{A.8})$$

A.4 Vector-Wilson 3

$$\begin{aligned} \mathcal{P}_{(1)\{\mu|\sigma\rho\lambda\}}^{V,W_3}(q) &= \eta_{\mu\sigma}\eta_{\rho\lambda} + \eta_{\mu\rho}\eta_{\sigma\lambda} + \eta_{\mu\lambda}\eta_{\rho\sigma} \\ &\quad - (d+2) [\eta_{\mu\sigma}q_\rho q_\lambda + \eta_{\mu\rho}q_\sigma q_\lambda + \eta_{\mu\lambda}q_\rho q_\sigma] \frac{1}{q^2} \\ &\quad + 2(d+2) \frac{q_\mu q_\sigma q_\rho q_\lambda}{(q^2)^2} \\ \mathcal{P}_{(2)\{\mu|\sigma\rho\lambda\}}^{V,W_3}(q) &= [\eta_{\sigma\rho}q_\mu q_\lambda + \eta_{\sigma\lambda}q_\mu q_\rho + \eta_{\rho\lambda}q_\mu q_\sigma] \frac{1}{q^2} - (d+2) \frac{q_\mu q_\sigma q_\rho q_\lambda}{(q^2)^2} \end{aligned} \quad (\text{A.9})$$

$$\mathcal{M}^{V,W_3} = \frac{1}{3(d-1)(d-1)(d+1)(q^2)^2} \begin{pmatrix} 1 & -1 \\ -1 & 3d+4 \end{pmatrix}. \quad (\text{A.10})$$

A.5 Wilson 2-Wilson 2

$$\begin{aligned} \mathcal{P}_{(1)\{\mu\nu|\sigma\rho\}}^{W_2,W_2}(q) &= \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{2}{d}\eta_{\mu\nu}\eta_{\sigma\rho} \\ \mathcal{P}_{(2)\{\mu\nu|\sigma\rho\}}^{W_2,W_2}(q) &= -\frac{1}{d}\eta_{\mu\nu}\eta_{\sigma\rho} + [\eta_{\mu\nu}q_\sigma q_\rho + \eta_{\sigma\rho}q_\mu q_\nu] \frac{1}{q^2} - d \frac{q_\mu q_\nu q_\sigma q_\rho}{(q^2)^2} \\ \mathcal{P}_{(3)\{\mu\nu|\sigma\rho\}}^{W_2,W_2}(q) &= \left[\eta_{\mu\sigma}q_\nu q_\rho + \eta_{\mu\rho}q_\nu q_\sigma + \eta_{\nu\sigma}q_\mu q_\rho + \eta_{\nu\rho}q_\mu q_\sigma - 4 \frac{q_\mu q_\nu q_\sigma q_\rho}{q^2} \right] \frac{1}{q^2} \end{aligned} \quad (\text{A.11})$$

$$\mathcal{M}^{W_2,W_2} = \frac{1}{4(d-1)(d+1)(d-2)(q^2)^2} \begin{pmatrix} 2(d-1) & 4 & -2(d-1) \\ 4 & 4d & -4 \\ -2(d-1) & -4 & (d^2+d-4) \end{pmatrix}. \quad (\text{A.12})$$

A.6 Wilson 3-Wilson 3

$$\begin{aligned} \mathcal{P}_{(1)\{\mu\nu\sigma|\rho\lambda\psi\}}^{W_3,W_3}(q) &= \eta_{\mu\nu}\eta_{\sigma\rho}\eta_{\lambda\psi} + \eta_{\mu\nu}\eta_{\sigma\lambda}\eta_{\rho\psi} + \eta_{\mu\nu}\eta_{\sigma\psi}\eta_{\rho\lambda} + \eta_{\mu\sigma}\eta_{\nu\rho}\eta_{\lambda\psi} + \eta_{\mu\sigma}\eta_{\nu\lambda}\eta_{\rho\psi} \\ &\quad + \eta_{\mu\sigma}\eta_{\nu\psi}\eta_{\rho\lambda} + \eta_{\mu\rho}\eta_{\nu\sigma}\eta_{\lambda\psi} + \eta_{\mu\lambda}\eta_{\nu\sigma}\eta_{\rho\psi} + \eta_{\mu\psi}\eta_{\nu\sigma}\eta_{\rho\lambda} \\ &\quad - \frac{(d+2)}{q^2} [\eta_{\mu\nu}\eta_{\sigma\rho}q_\lambda q_\psi + \eta_{\mu\nu}\eta_{\sigma\lambda}q_\rho q_\psi + \eta_{\mu\nu}\eta_{\sigma\psi}q_\lambda q_\rho \\ &\quad + \eta_{\mu\sigma}\eta_{\nu\rho}q_\lambda q_\psi + \eta_{\mu\sigma}\eta_{\nu\lambda}q_\rho q_\psi + \eta_{\mu\sigma}\eta_{\nu\psi}q_\lambda q_\rho \\ &\quad + \eta_{\mu\rho}\eta_{\nu\sigma}q_\lambda q_\psi + \eta_{\mu\rho}\eta_{\nu\lambda}\psi q_\sigma + \eta_{\mu\lambda}\eta_{\nu\sigma}q_\rho q_\psi \\ &\quad + \eta_{\mu\lambda}\eta_{\rho\psi}q_\nu q_\sigma + \eta_{\mu\psi}\eta_{\nu\sigma}q_\rho q_\lambda + \eta_{\mu\psi}\eta_{\rho\lambda}q_\nu q_\sigma \\ &\quad + \eta_{\nu\rho}\eta_{\lambda\psi}q_\mu q_\sigma + \eta_{\nu\lambda}\eta_{\rho\psi}q_\mu q_\sigma + \eta_{\nu\psi}\eta_{\rho\lambda}q_\mu q_\sigma \\ &\quad + \eta_{\sigma\rho}\eta_{\lambda\psi}q_\mu q_\nu + \eta_{\sigma\lambda}\eta_{\rho\psi}q_\mu q_\nu + \eta_{\sigma\psi}\eta_{\rho\lambda}q_\mu q_\nu] \\ &\quad + \frac{2(d+2)}{(q^2)^2} [\eta_{\mu\nu}q_\sigma q_\rho q_\lambda q_\psi + \eta_{\mu\sigma}q_\nu q_\rho q_\lambda q_\psi + \eta_{\nu\sigma}q_\mu q_\rho q_\lambda q_\psi \\ &\quad + \eta_{\rho\lambda}q_\mu q_\nu q_\sigma q_\psi + \eta_{\rho\psi}q_\mu q_\nu q_\sigma q_\lambda + \eta_{\lambda\psi}q_\mu q_\nu q_\sigma q_\rho] \end{aligned}$$

$$\begin{aligned}
 & + \frac{(d+2)^2}{(q^2)^2} [\eta_{\mu\rho}q_\nu q_\sigma q_\lambda q_\psi + \eta_{\mu\lambda}q_\nu q_\sigma q_\rho q_\psi + \eta_{\mu\psi}q_\nu q_\sigma q_\rho q_\lambda \\
 & + \eta_{\nu\rho}q_\mu q_\sigma q_\lambda q_\psi + \eta_{\nu\lambda}q_\mu q_\sigma q_\rho q_\psi + \eta_{\nu\psi}q_\mu q_\sigma q_\rho q_\lambda \\
 & + \eta_{\sigma\rho}q_\mu q_\nu q_\lambda q_\psi + \eta_{\sigma\lambda}q_\mu q_\nu q_\rho q_\psi + \eta_{\sigma\psi}q_\mu q_\nu q_\rho q_\lambda] \\
 & - \frac{8(d+2)^2}{(q^2)^3} q_\mu q_\nu q_\sigma q_\rho q_\lambda q_\psi \\
 \mathcal{P}_{(2)\{\mu\nu\sigma|\rho\lambda\psi\}}^{W_3, W_3}(q) = & \eta_{\mu\rho}\eta_{\nu\lambda}\eta_{\sigma\psi} + \eta_{\mu\rho}\eta_{\nu\psi}\eta_{\sigma\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho}\eta_{\sigma\psi} \\
 & + \eta_{\mu\lambda}\eta_{\nu\psi}\eta_{\sigma\rho} + \eta_{\mu\psi}\eta_{\nu\rho}\eta_{\sigma\lambda} + \eta_{\mu\psi}\eta_{\nu\lambda}\eta_{\sigma\rho} \\
 & - \frac{2}{q^2} [\eta_{\mu\nu}\eta_{\sigma\rho}q_\lambda q_\psi + \eta_{\mu\nu}\eta_{\sigma\lambda}q_\rho q_\psi + \eta_{\mu\nu}\eta_{\sigma\psi}q_\lambda q_\rho \\
 & + \eta_{\mu\sigma}\eta_{\nu\rho}q_\lambda q_\psi + \eta_{\mu\sigma}\eta_{\nu\lambda}q_\rho q_\psi + \eta_{\mu\sigma}\eta_{\nu\psi}q_\lambda q_\rho \\
 & + \eta_{\mu\rho}\eta_{\nu\sigma}q_\lambda q_\psi + \eta_{\mu\rho}\eta_{\lambda\psi}q_\nu q_\sigma + \eta_{\mu\lambda}\eta_{\nu\sigma}q_\rho q_\psi \\
 & + \eta_{\mu\lambda}\eta_{\rho\psi}q_\nu q_\sigma + \eta_{\mu\psi}\eta_{\nu\sigma}q_\rho q_\lambda + \eta_{\mu\psi}\eta_{\rho\lambda}q_\nu q_\sigma \\
 & + \eta_{\nu\rho}\eta_{\lambda\psi}q_\mu q_\sigma + \eta_{\nu\lambda}\eta_{\rho\psi}q_\mu q_\sigma + \eta_{\nu\psi}\eta_{\rho\lambda}q_\mu q_\sigma \\
 & + \eta_{\sigma\rho}\eta_{\lambda\psi}q_\mu q_\nu + \eta_{\sigma\lambda}\eta_{\rho\psi}q_\mu q_\nu + \eta_{\sigma\psi}\eta_{\rho\lambda}q_\mu q_\nu] \\
 & + \frac{4}{(q^2)^2} [\eta_{\mu\nu}q_\sigma q_\rho q_\lambda q_\psi + \eta_{\mu\sigma}q_\nu q_\rho q_\lambda q_\psi + \eta_{\nu\sigma}q_\mu q_\rho q_\lambda q_\psi \\
 & + \eta_{\rho\lambda}q_\mu q_\nu q_\sigma q_\psi + \eta_{\rho\psi}q_\mu q_\nu q_\sigma q_\lambda + \eta_{\lambda\psi}q_\mu q_\nu q_\sigma q_\rho] \\
 & + \frac{2(d+2)}{(q^2)^2} [\eta_{\mu\rho}q_\nu q_\sigma q_\lambda q_\psi + \eta_{\mu\lambda}q_\nu q_\sigma q_\rho q_\psi + \eta_{\mu\psi}q_\nu q_\sigma q_\rho q_\lambda \\
 & + \eta_{\nu\rho}q_\mu q_\sigma q_\lambda q_\psi + \eta_{\nu\lambda}q_\mu q_\sigma q_\rho q_\psi + \eta_{\nu\psi}q_\mu q_\sigma q_\rho q_\lambda \\
 & + \eta_{\sigma\rho}q_\mu q_\nu q_\lambda q_\psi + \eta_{\sigma\lambda}q_\mu q_\nu q_\rho q_\psi + \eta_{\sigma\psi}q_\mu q_\nu q_\rho q_\lambda] \\
 & - \frac{16(d+2)}{(q^2)^3} q_\mu q_\nu q_\sigma q_\rho q_\lambda q_\psi \\
 \mathcal{P}_{(3)\{\mu\nu\sigma|\rho\lambda\psi\}}^{W_3, W_3}(q) = & \frac{1}{q^2} [\eta_{\mu\nu}\eta_{\rho\lambda}q_\sigma q_\psi + \eta_{\mu\nu}\eta_{\rho\psi}q_\sigma q_\lambda + \eta_{\mu\nu}\eta_{\lambda\psi}q_\sigma q_\rho \\
 & + \eta_{\mu\sigma}\eta_{\rho\lambda}q_\nu q_\psi + \eta_{\mu\sigma}\eta_{\rho\psi}q_\nu q_\lambda + \eta_{\mu\sigma}\eta_{\lambda\psi}q_\nu q_\rho \\
 & + \eta_{\nu\sigma}\eta_{\rho\lambda}q_\mu q_\psi + \eta_{\nu\sigma}\eta_{\rho\psi}q_\mu q_\lambda + \eta_{\nu\sigma}\eta_{\lambda\psi}q_\mu q_\rho] \\
 & - \frac{(d+2)}{(q^2)^2} [\eta_{\mu\nu}q_\sigma q_\rho q_\lambda q_\psi + \eta_{\mu\sigma}q_\nu q_\rho q_\lambda q_\psi + \eta_{\nu\sigma}q_\mu q_\rho q_\lambda q_\psi \\
 & + \eta_{\rho\lambda}q_\mu q_\nu q_\sigma q_\psi + \eta_{\rho\psi}q_\mu q_\nu q_\sigma q_\lambda + \eta_{\lambda\psi}q_\mu q_\nu q_\sigma q_\rho] \\
 & + \frac{(d+2)^2}{(q^2)^3} q_\mu q_\nu q_\sigma q_\rho q_\lambda q_\psi \\
 \mathcal{P}_{(4)\{\mu\nu\sigma|\rho\lambda\psi\}}^{W_3, W_3}(q) = & \frac{1}{q^2} [\eta_{\mu\rho}\eta_{\nu\lambda}q_\sigma q_\psi + \eta_{\mu\rho}\eta_{\nu\psi}q_\sigma q_\lambda + \eta_{\mu\rho}\eta_{\sigma\lambda}q_\nu q_\psi \\
 & + \eta_{\mu\rho}\eta_{\sigma\psi}q_\nu q_\lambda + \eta_{\mu\lambda}\eta_{\nu\rho}q_\sigma q_\psi + \eta_{\mu\lambda}\eta_{\nu\psi}q_\sigma q_\rho \\
 & + \eta_{\mu\lambda}\eta_{\sigma\rho}q_\nu q_\psi + \eta_{\mu\lambda}\eta_{\sigma\psi}q_\nu q_\rho + \eta_{\mu\psi}\eta_{\nu\rho}q_\sigma q_\lambda \\
 & + \eta_{\mu\psi}\eta_{\nu\lambda}q_\sigma q_\rho + \eta_{\mu\psi}\eta_{\sigma\rho}q_\nu q_\lambda + \eta_{\mu\psi}\eta_{\sigma\lambda}q_\nu q_\rho \\
 & + \eta_{\nu\rho}\eta_{\sigma\lambda}q_\mu q_\psi + \eta_{\nu\rho}\eta_{\sigma\psi}q_\mu q_\lambda + \eta_{\nu\lambda}\eta_{\sigma\rho}q_\mu q_\psi \\
 & + \eta_{\nu\lambda}\eta_{\sigma\psi}q_\mu q_\rho + \eta_{\nu\psi}\eta_{\sigma\rho}q_\mu q_\lambda + \eta_{\nu\psi}\eta_{\sigma\lambda}q_\mu q_\rho]
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{2}{(q^2)^2} [\eta_{\mu\nu} q_\sigma q_\rho q_\lambda q_\psi + \eta_{\mu\sigma} q_\nu q_\rho q_\lambda q_\psi + \eta_{\nu\sigma} q_\mu q_\rho q_\lambda q_\psi \\
 & + \eta_{\rho\lambda} q_\mu q_\nu q_\sigma q_\psi + \eta_{\rho\psi} q_\mu q_\nu q_\sigma q_\lambda + \eta_{\lambda\psi} q_\mu q_\nu q_\sigma q_\rho] \\
 & - \frac{4}{(q^2)^2} [\eta_{\mu\rho} q_\nu q_\sigma q_\lambda q_\psi + \eta_{\mu\lambda} q_\nu q_\sigma q_\rho q_\psi + \eta_{\mu\psi} q_\nu q_\sigma q_\rho q_\lambda \\
 & + \eta_{\nu\rho} q_\mu q_\sigma q_\lambda q_\psi + \eta_{\nu\lambda} q_\mu q_\sigma q_\rho q_\psi + \eta_{\nu\psi} q_\mu q_\sigma q_\rho q_\lambda \\
 & + \eta_{\sigma\rho} q_\mu q_\nu q_\lambda q_\psi + \eta_{\sigma\lambda} q_\mu q_\nu q_\rho q_\psi + \eta_{\sigma\psi} q_\mu q_\nu q_\rho q_\lambda] \\
 & + \frac{2(d+14)}{(q^2)^3} q_\mu q_\nu q_\sigma q_\rho q_\lambda q_\psi
 \end{aligned} \tag{A.13}$$

$$\mathcal{M}^{W_3, W_3} = \frac{1}{18(d^2-1)(d-2)(d+2)^2(d+3)(q^2)^3} \times \begin{pmatrix} 2(7d+18) & -6(d+2)^2 & -2(7d+18) & 6(d+2)^2 \\ -6(d+2)^2 & 3(d+1)(d+2)^2 & 6(d+2)^2 & -3(d+1)(d+2)^2 \\ -2(7d+18) & 6(d+2)^2 & 2(11d^2+50d+48) & -2(d+6)(d+2)^2 \\ 6(d+2)^2 & -3(d+1)(d+2)^2 & -2(d+6)(d+2)^2 & d(d+5)(d+2)^2 \end{pmatrix} \tag{A.14}$$

A.7 Tensor-transversity 2

$$\begin{aligned}
 \mathcal{P}_{(1)\{\mu\nu|\sigma\rho\lambda\}}^{T, T_2}(q) &= \eta_{\mu\sigma} \eta_{\nu\rho} q_\lambda + \eta_{\mu\sigma} \eta_{\nu\lambda} q_\rho - \eta_{\mu\rho} \eta_{\nu\sigma} q_\lambda - \eta_{\mu\lambda} \eta_{\nu\sigma} q_\rho \\
 &+ [\eta_{\mu\rho} q_\nu q_\sigma q_\lambda + \eta_{\mu\lambda} q_\nu q_\sigma q_\rho - \eta_{\nu\rho} q_\mu q_\sigma q_\lambda - \eta_{\nu\lambda} q_\mu q_\sigma q_\rho \\
 &+ 2\eta_{\nu\sigma} q_\mu q_\rho q_\lambda - 2\eta_{\mu\sigma} q_\nu q_\rho q_\lambda] \frac{1}{q^2} \\
 \mathcal{P}_{(2)\{\mu\nu|\sigma\rho\lambda\}}^{T, T_2}(q) &= \eta_{\mu\sigma} \eta_{\rho\lambda} q_\nu - \eta_{\nu\sigma} \eta_{\rho\lambda} q_\mu \\
 &+ [\eta_{\nu\rho} q_\mu q_\sigma q_\lambda + \eta_{\nu\lambda} q_\mu q_\sigma q_\rho - \eta_{\mu\rho} q_\nu q_\sigma q_\lambda - \eta_{\mu\lambda} q_\nu q_\sigma q_\rho \\
 &+ d\eta_{\nu\sigma} q_\mu q_\rho q_\lambda - d\eta_{\mu\sigma} q_\nu q_\rho q_\lambda] \frac{1}{q^2} \\
 \mathcal{P}_{(3)\{\mu\nu|\sigma\rho\lambda\}}^{T, T_2}(q) &= \eta_{\mu\rho} \eta_{\sigma\lambda} q_\nu + \eta_{\mu\lambda} \eta_{\sigma\rho} q_\nu - \eta_{\nu\rho} \eta_{\sigma\lambda} q_\nu - \eta_{\nu\lambda} \eta_{\sigma\rho} q_\mu \\
 &+ [(d+1)(\eta_{\nu\rho} q_\mu q_\sigma q_\lambda + \eta_{\nu\lambda} q_\mu q_\sigma q_\rho - \eta_{\mu\rho} q_\nu q_\sigma q_\lambda - \eta_{\mu\lambda} q_\nu q_\sigma q_\rho) \\
 &+ 2\eta_{\nu\sigma} q_\mu q_\rho q_\lambda - 2\eta_{\mu\sigma} q_\nu q_\rho q_\lambda] \frac{1}{q^2} \\
 \mathcal{P}_{(4)\{\mu\nu|\sigma\rho\lambda\}}^{T, T_2}(q) &= \eta_{\mu\rho} \eta_{\nu\lambda} q_\sigma - \eta_{\mu\lambda} \eta_{\nu\rho} q_\sigma \\
 &+ [\eta_{\nu\rho} q_\mu q_\sigma q_\lambda + \eta_{\nu\lambda} q_\mu q_\sigma q_\rho - \eta_{\mu\rho} q_\nu q_\sigma q_\lambda - \eta_{\mu\lambda} q_\nu q_\sigma q_\rho] \frac{1}{q^2}
 \end{aligned} \tag{A.15}$$

$$\mathcal{M}^{T, T_2} = \frac{1}{4d^2(d^2-1)(d-2)(q^2)^2} \begin{pmatrix} d^2(d+1) & 0 & 0 & 0 \\ 0 & 2(d^2+4) & -4d & 4(d-2) \\ 0 & -4d & d^2 & -2d(d-2) \\ 0 & 4(d-2) & -2d(d-2) & 2(d-1)(d^2-4) \end{pmatrix}. \tag{A.16}$$

B Expressions for SU(3)

For completeness and for practical use, we record the explicit numerical values of the various amplitudes for the colour group SU(3). We take the usual values for the Casimirs, $T_F = \frac{1}{2}$, $C_A = 3$ and $C_F = \frac{4}{3}$ as well as $d(R) = 3$ but leave the numbers of quarks unfixed. We only record those amplitudes which are non-zero. The remaining ones still satisfy the same relations to third order which were noted in section 4. Thus, we have

B.1 Vector currents

$$\begin{aligned}
 \Pi^{S,S}(a) &= 3 \left[4.000000 + 2.000000\ell + \left[48.867512 + 45.333333\ell + 8.000000\ell^2 \right] a \right. \\
 &\quad + \left[1925.894130 - 100.119646N_f + (1650.138715 - 61.022786N_f)\ell \right. \\
 &\quad + (565.333333 - 19.555555N_f)\ell^2 \\
 &\quad \left. \left. + (50.666667 - 1.777778N_f)\ell^3 \right] a^2 \right] + O(a^3) \\
 \Pi_{(1)}^{V,V}(a) &= 3 \left[-2.222222 - 1.333333\ell + \left[1.199436 - 5.333333\ell \right] a \right. \\
 &\quad + \left[6.784729N_f - 38.534112 + (2.459635N_f - 42.361758)\ell \right. \\
 &\quad \left. \left. + (1.777778N_f - 29.333333)\ell^2 \right] a^2 \right] + O(a^3) \\
 \Pi_{(1)}^{T,T}(a) &= 3 \left[-0.444444 - 0.666667\ell + \left[0.698484 - 2.074074\ell + 0.888889\ell^2 \right] a \right. \\
 &\quad + \left[27.577316 - 1.114284N_f + (22.743851 - 2.243433N_f)\ell \right. \\
 &\quad + (18.172840 - 0.592593N_f)\ell^2 \\
 &\quad \left. \left. + (2.469136 - 0.197531N_f)\ell^3 \right] a^2 \right] + O(a^3) \\
 \Pi_{(2)}^{T,T}(a) &= 3 \left[2.222222 + 1.333333\ell + \left[3.640070 + 4.148148\ell - 1.777778\ell^2 \right] a \right. \\
 &\quad + \left[-32.988175 + 2.272463N_f + (-45.487702 + 4.486867N_f)\ell \right. \\
 &\quad + (-36.345679 + 1.185185N_f)\ell^2 \\
 &\quad \left. \left. + (-4.938272 + 0.395062N_f)\ell^3 \right] a^2 \right] + O(a^3) \tag{B.1}
 \end{aligned}$$

B.2 Wilson moment $n = 2$

$$\begin{aligned}
 \Pi_{(1)}^{W_2,W_2}(a) &= 3 \left[0.480000 + 0.200000\ell + \left[-5.578236 - 2.802963\ell - 0.711111\ell^2 \right] a \right. \\
 &\quad + \left[-88.800839 + 11.700261N_f + (-56.861337 + 7.125112N_f)\ell \right. \\
 &\quad + (-15.116049 + 1.566420N_f)\ell^2 \\
 &\quad \left. \left. + (-0.921811 + 0.158025N_f)\ell^3 \right] a^2 \right] + O(a^3) \\
 \Pi_{(2)}^{W_2,W_2}(a) &= 3 \left[0.408889 + 0.133333\ell + \left[-3.597014 - 1.947654\ell - 0.474074\ell^2 \right] a \right. \\
 &\quad + \left[-57.728474 + 8.008477N_f + (-39.368555 + 4.896687N_f)\ell \right. \\
 &\quad + (-10.231001 + 1.070617N_f)\ell^2 \\
 &\quad \left. \left. + (-0.614540 + 0.105350N_f)\ell^3 \right] a^2 \right] + O(a^3) \\
 \Pi_{(3)}^{W_2,W_2}(a) &= 3 \left[0.075555 + 0.133333\ell + \left[4.238377 + 3.780741\ell + 0.711111\ell^2 \right] a \right. \\
 &\quad + \left[75.027286 - 10.992767N_f + (56.696221 - 6.724712N_f)\ell \right. \\
 &\quad \left. \left. + (21.758025 - 1.892346N_f)\ell^2 \right] a^2 \right] + O(a^3)
 \end{aligned}$$

$$\begin{aligned}
& + (0.921811 - 0.158025N_f) \ell^3] a^2] + O(a^3) \\
\Pi_{(3)}^{W_2, \partial W_2}(a) = & 3 [1.111111 + 0.666667\ell + [-0.599718 + 2.666667\ell] a \\
& + [19.267056 - 3.392365N_f + (21.180879 - 1.229818N_f)\ell \\
& + (14.666667 - 0.888889N_f)\ell^2] a^2] + O(a^3) \tag{B.2}
\end{aligned}$$

B.3 Wilson moment $n = 3$

$$\begin{aligned}
\Pi_{(1)}^{V, W_3}(a) = & 3 [0.045926 + 0.022222\ell + [0.020544 + 0.098765\ell] a \\
& + [0.917043 - 0.131672N_f + (0.838448 - 0.047350N_f)\ell \\
& + (0.497942 - 0.030178N_f)\ell^2] a^2] + O(a^3) \\
\Pi_{(1)}^{V, \partial W_3}(a) = & 3 [0.074074 + 0.037037\ell + [-0.008626 + 0.148148\ell] a \\
& + [1.211681 - 0.196695N_f + (1.176715 - 0.068323N_f)\ell \\
& + (0.814815 - 0.049383N_f)\ell^2] a^2] + O(a^3) \tag{B.3} \\
\Pi_{(1)}^{W_3, W_3}(a) = & 3 [0.001151 + 0.000265\ell + [-0.037289 - 0.020408\ell - 0.003527\ell^2] a \\
& + [-0.344878 + 0.084926N_f + (-0.203993 + 0.046931N_f)\ell \\
& + (-0.050228 + 0.009964N_f)\ell^2 \\
& + (0.000131 + 0.000784N_f)\ell^3] a^2] + O(a^3) \\
\Pi_{(2)}^{W_3, W_3}(a) = & 3 [-0.004528 - 0.001587\ell + [0.100640 + 0.049139\ell + 0.008818\ell^2] a \\
& + [1.022184 - 0.224027N_f + (0.551061 - 0.122691N_f)\ell \\
& + (0.109786 - 0.024512N_f)\ell^2 \\
& + (-0.000327 - 0.001960N_f)\ell^3] a^2] + O(a^3) \\
\Pi_{(3)}^{W_3, W_3}(a) = & 3 [0.010207 + 0.003439\ell + [-0.069547 - 0.035555\ell - 0.009641\ell^2] a \\
& + [-1.354970 + 0.153442N_f + (-0.921737 + 0.095085N_f)\ell \\
& + (-0.233863 + 0.020396N_f)\ell^2 \\
& + (-0.017201 + 0.002143N_f)\ell^3] a^2] + O(a^3) \\
\Pi_{(4)}^{W_3, W_3}(a) = & 3 [-0.008806 - 0.003968\ell + [0.063950 + 0.031660\ell + 0.010935\ell^2] a \\
& + [1.580679 - 0.124987N_f + (1.078542 - 0.084549N_f)\ell \\
& + (0.309941 - 0.019979N_f)\ell^2 \\
& + (0.025932 - 0.002430N_f)\ell^3] a^2] + O(a^3) \\
\Pi_{(1)}^{W_3, \partial W_3}(a) = & 3 [-0.001481 - 0.000617\ell + [0.002510 - 0.001641\ell] a \\
& + [0.048778 - 0.003811N_f + (0.009906 - 0.001818N_f)\ell \\
& + (-0.011698 + 0.000472N_f)\ell^2] a^2] + O(a^3) \\
\Pi_{(3)}^{W_3, \partial W_3}(a) = & 3 [0.012840 + 0.004321\ell + [-0.102610 - 0.052455\ell - 0.013169\ell^2] a \\
& + [-1.653254 + 0.226264N_f + (-1.103477 + 0.137837N_f)\ell \\
& + (-0.272497 + 0.029267N_f)\ell^2 \\
& + (-0.017071 + 0.002926N_f)\ell^3] a^2] + O(a^3) \\
\Pi_{(4)}^{W_3, \partial W_3}(a) = & 3 [-0.013333 - 0.005555\ell + [0.154951 + 0.077860\ell + 0.019753\ell^2] a
\end{aligned}$$

$$\begin{aligned}
 & + [2.466690 - 0.325007N_f + (1.579482 - 0.197920N_f)\ell \\
 & + (0.419890 - 0.043512N_f)\ell^2 \\
 & + (0.025606 - 0.004390N_f)\ell^3] a^2] + O(a^3) \\
 \Pi_{(1)}^{W_3, \partial\partial W_3}(a) & = 3[-0.002963 - 0.001235\ell + [-0.002132 - 0.005487\ell] a \\
 & + [-0.055664 + 0.007575N_f + (-0.046580 + 0.002631N_f)\ell \\
 & + (-0.027663 + 0.001677N_f)\ell^2] a^2] + O(a^3) \\
 \Pi_{(1)}^{\partial W_3, \partial W_3}(a) & = 3[-0.002401 - 0.001029\ell + [0.003129 - 0.003018\ell] a \\
 & + [0.039866 - 0.001957N_f + (0.000510 - 0.001236N_f)\ell \\
 & + (-0.020450 + 0.001006N_f)\ell^2] a^2] + O(a^3) \\
 \Pi_{(3)}^{\partial W_3, \partial W_3}(a) & = 3[0.013759 + 0.004733\ell + [-0.103229 - 0.051084\ell - 0.013169\ell^2] a \\
 & + [-1.644342 + 0.224410N_f + (-1.094081 + 0.137255N_f)\ell \\
 & + (-0.263695 + 0.028733N_f)\ell^2 \\
 & + (-0.017071 + 0.002926N_f)\ell^3] a^2] + O(a^3) \\
 \Pi_{(1)}^{\partial W_3, \partial\partial W_3}(a) & = 3[-0.004801 - 0.002058\ell + [-0.000893 - 0.008230\ell] a \\
 & + [-0.073488 + 0.011283N_f + (-0.065373 + 0.003796N_f)\ell \\
 & + (-0.045267 + 0.002743N_f)\ell^2] a^2] + O(a^3) \tag{B.4}
 \end{aligned}$$

B.4 Transversity moment $n = 2$

$$\begin{aligned}
 \Pi_{(1)}^{T, T_2}(a) & = 3[0.222222 + 0.333333\ell + [-0.349242 + 1.037037\ell - 0.444444\ell^2] a \\
 & + [-13.788658 + 0.557142N_f + (-11.371925 + 1.121717N_f)\ell \\
 & + (-9.086420 + 0.296296N_f)\ell^2 \\
 & + (-1.234568 + 0.098765N_f)\ell^3] a^2] + O(a^3) \\
 \Pi_{(2)}^{T, T_2}(a) & = 3[0.740741 + 0.222222\ell + [2.335073 + 0.691358\ell - 0.296296\ell^2] a \\
 & + [5.429324 + 0.264127N_f + (-7.581284 + 0.747811N_f)\ell \\
 & + (-6.057613 + 0.197531N_f)\ell^2 \\
 & + (-0.823045 + 0.065844N_f)\ell^3] a^2] + O(a^3) . \tag{B.5}
 \end{aligned}$$

References

- [1] D.J. Gross and F.J. Wilczek, *Asymptotically free gauge theories. 2*, *Phys. Rev. D* **9** (1974) 980 [SPIRES].
- [2] E.G. Floratos, D.A. Ross and C.T. Sachrajda, *Higher order effects in asymptotically free gauge theories: the anomalous dimensions of Wilson operators*, *Nucl. Phys. B* **129** (1977) 66 [Erratum *ibid.* **B139** (1978) 545] [SPIRES].
- [3] E.G. Floratos, D.A. Ross and C.T. Sachrajda, *Higher order effects in asymptotically free gauge theories. 2. Flavor singlet Wilson operators and coefficient functions*, *Nucl. Phys. B* **152** (1979) 493 [SPIRES].
- [4] S. Moch, J.A.M. Vermaseren and A. Vogt, *The three-loop splitting functions in QCD: the non-singlet case*, *Nucl. Phys. B* **688** (2004) 101 [hep-ph/0403192] [SPIRES].

- [5] A. Vogt, S. Moch and J.A.M. Vermaseren, *The three-loop splitting functions in QCD: the singlet case*, *Nucl. Phys. B* **691** (2004) 129 [[hep-ph/0404111](#)] [[SPIRES](#)].
- [6] S. Moch, J.A.M. Vermaseren and A. Vogt, *The QCD splitting functions at three loops: methods and results*, *Nucl. Phys. Proc. Suppl.* **135** (2004) 137 [[hep-ph/0408075](#)] [[SPIRES](#)].
- [7] S. Moch, J.A.M. Vermaseren and A. Vogt, *The longitudinal structure function at the third order*, *Phys. Lett. B* **606** (2005) 123 [[hep-ph/0411112](#)] [[SPIRES](#)].
- [8] M. Gockeler et al., *Calculation of moments of structure functions*, *Nucl. Phys. Proc. Suppl.* **119** (2003) 32 [[hep-lat/0209160](#)] [[SPIRES](#)].
- [9] M. Gockeler et al., *Nonperturbative renormalisation of composite operators in lattice QCD*, *Nucl. Phys. B* **544** (1999) 699 [[hep-lat/9807044](#)] [[SPIRES](#)].
- [10] S. Capitani et al., *Renormalisation and off-shell improvement in lattice perturbation theory*, *Nucl. Phys. B* **593** (2001) 183 [[hep-lat/0007004](#)] [[SPIRES](#)].
- [11] C. Gatttringer, M. Gockeler, P. Huber and C.B. Lang, *Renormalization of bilinear quark operators for the chirally improved lattice Dirac operator*, *Nucl. Phys. B* **694** (2004) 170 [[hep-lat/0404006](#)] [[SPIRES](#)].
- [12] QCDSF collaboration, M. Gockeler, R. Horsley, D. Pleiter, P.E.L. Rakow and G. Schierholz, *A lattice determination of moments of unpolarised nucleon structure functions using improved Wilson fermions*, *Phys. Rev. D* **71** (2005) 114511 [[hep-ph/0410187](#)] [[SPIRES](#)].
- [13] M. Gürtler, R. Horsley, P.E.L. Rakow, C.J. Roberts, G. Schierholz and T. Streuer, *Non-perturbative renormalisation for overlap fermions*, *PoS(LAT2005)125* [[hep-lat/0510045](#)] [[SPIRES](#)].
- [14] CSSM LATTICE collaboration, J.B. Zhang, D.B. Leinweber, K.F. Liu and A.G. Williams, *Nonperturbative renormalisation of composite operators with overlap quarks*, *Nucl. Phys. Proc. Suppl.* **128** (2004) 240 [[hep-lat/0311030](#)] [[SPIRES](#)].
- [15] D. Becirevic et al., *Renormalization constants of quark operators for the non-perturbatively improved Wilson action*, *JHEP* **08** (2004) 022 [[hep-lat/0401033](#)] [[SPIRES](#)].
- [16] J.B. Zhang et al., *Nonperturbative renormalization of composite operators with overlap fermions*, *Phys. Rev. D* **72** (2005) 114509 [[hep-lat/0507022](#)] [[SPIRES](#)].
- [17] F. Di Renzo, A. Mantovi, V. Miccio, C. Torrero and L. Scorzato, *Wilson fermions quark bilinears to three loops*, *PoS(LAT2005)237* [[hep-lat/0509158](#)] [[SPIRES](#)].
- [18] G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa and A. Vladikas, *A general method for nonperturbative renormalization of lattice operators*, *Nucl. Phys. B* **445** (1995) 81 [[hep-lat/9411010](#)] [[SPIRES](#)].
- [19] E. Franco and V. Lubicz, *Quark mass renormalization in the \overline{MS} and RI schemes up to the NNLO order*, *Nucl. Phys. B* **531** (1998) 641 [[hep-ph/9803491](#)] [[SPIRES](#)].
- [20] K.G. Chetyrkin and A. Retey, *Renormalization and running of quark mass and field in the regularization invariant and \overline{MS} schemes at three and four loops*, *Nucl. Phys. B* **583** (2000) 3 [[hep-ph/9910332](#)] [[SPIRES](#)].
- [21] J.A. Gracey, *Three loop anomalous dimension of non-singlet quark currents in the RI scheme*, *Nucl. Phys. B* **662** (2003) 247 [[hep-ph/0304113](#)] [[SPIRES](#)].
- [22] J.A. Gracey, *Three loop anomalous dimension of the second moment of the transversity operator in the \overline{MS} and RI schemes*, *Nucl. Phys. B* **667** (2003) 242 [[hep-ph/0306163](#)] [[SPIRES](#)].

- [23] J.A. Gracey, *Three loop anomalous dimensions of higher moments of the non-singlet twist-2 Wilson and transversity operators in the \overline{MS} and R' schemes*, *JHEP* **10** (2006) 040 [[hep-ph/0609231](#)] [[SPIRES](#)].
- [24] M. Lüscher, R. Narayanan, P. Weisz and U. Wolff, *The Schrödinger functional: a renormalizable probe for nonAbelian gauge theories*, *Nucl. Phys. B* **384** (1992) 168 [[hep-lat/9207009](#)] [[SPIRES](#)].
- [25] ZEUTHEN-ROME/ZERO collaboration, M. Guagnelli et al., *Continuous external momenta in non-perturbative lattice simulations: a computation of renormalization factors*, *Nucl. Phys. B* **664** (2003) 276 [[hep-lat/0303012](#)] [[SPIRES](#)].
- [26] K.G. Chetyrkin, J.H. Kuhn and A. Kwiatkowski, *QCD corrections to the e^+e^- cross-section and the Z boson decay rate*, [hep-ph/9503396](#) [[SPIRES](#)].
- [27] K.G. Chetyrkin, *Correlator of the quark scalar currents and $\Gamma_{tot}(H \rightarrow \text{hadrons})$ at $O(\alpha_s^3)$ in $pQCD$* , *Phys. Lett. B* **390** (1997) 309 [[hep-ph/9608318](#)] [[SPIRES](#)].
- [28] K.G. Chetyrkin, J.H. Kuhn and A. Kwiatkowski, *QCD corrections to the e^+e^- cross-section and the Z boson decay rate: concepts and results*, *Phys. Rept.* **277** (1996) 189 [[SPIRES](#)].
- [29] J.P. Ralston and D.E. Soper, *Production of dimuons from high-energy polarized proton proton collisions*, *Nucl. Phys. B* **152** (1979) 109 [[SPIRES](#)].
- [30] R.L. Jaffe and X.-D. Ji, *Chiral odd parton distributions and polarized Drell-Yan*, *Phys. Rev. Lett.* **67** (1991) 552 [[SPIRES](#)]; *Chiral odd parton distributions and Drell-Yan processes*, *Nucl. Phys. B* **375** (1992) 527 [[SPIRES](#)].
- [31] J.L. Cortes, B. Pire and J.P. Ralston, *Measuring the transverse polarization of quarks in the proton*, *Z. Phys. C* **55** (1992) 409 [[SPIRES](#)].
- [32] S.G. Gorishnii, S.A. Larin, L.R. Surguladze and F.V. Tkachov, *Mincer: Program for multiloop calculations in quantum field theory for the Schoonschip system*, *Comput. Phys. Commun.* **55** (1989) 381 [[SPIRES](#)].
- [33] J.A.M. Vermaseren, *New features of FORM*, [math-ph/0010025](#) [[SPIRES](#)].
- [34] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, *The form version of Mincer*, [NIKHEF-H-91-018](#).
- [35] P. Nogueira, *Automatic Feynman graph generation*, *J. Comput. Phys.* **105** (1993) 279 [[SPIRES](#)].
- [36] S.A. Larin and J.A.M. Vermaseren, *The three loop QCD β -function and anomalous dimensions*, *Phys. Lett. B* **303** (1993) 334 [[hep-ph/9302208](#)] [[SPIRES](#)].
- [37] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, *The next next-to-leading QCD approximation for nonsinglet moments of deep inelastic structure functions*, *Nucl. Phys. B* **427** (1994) 41 [[SPIRES](#)].
- [38] S.A. Larin, P. Nogueira, T. van Ritbergen and J.A.M. Vermaseren, *The 3-loop QCD calculation of the moments of deep inelastic structure functions*, *Nucl. Phys. B* **492** (1997) 338 [[hep-ph/9605317](#)] [[SPIRES](#)].
- [39] D.J. Broadhurst and A.G. Grozin, *Matching QCD and HQET heavy-light currents at two loops and beyond*, *Phys. Rev. D* **52** (1995) 4082 [[hep-ph/9410240](#)] [[SPIRES](#)].
- [40] D.J. Broadhurst, *Four-loop Dyson-Schwinger-Johnson anatomy*, *Phys. Lett. B* **466** (1999) 319 [[hep-ph/9909336](#)] [[SPIRES](#)].
- [41] J.A. Gracey, *Three loop \overline{MS} tensor current anomalous dimension in QCD*, *Phys. Lett. B* **488** (2000) 175 [[hep-ph/0007171](#)] [[SPIRES](#)].